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PROJECT LINCOLN

SOME MATHEMATICAL REMARKS ON THE BOOLEAN MACHINE

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19 December 1951

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ABSTRACT

This Report is presented in two parts: Part I comprises Sections 1 and 2; Part II consists of Sections 3 through 5.

In Section 1, a calculus for a Boolean function of a real variable is developed and utilized in the formation of a model of the simple Boolean machine. In Section 2, rules of probability are developed for Boolean events, and applied to the simple Boolean machine.

In Section 3, a canonical representation of the Boolean system is developed and discussed. Probability is introduced to the canonical representation of the simple Boolean system in the following Section, with a demonstration that the simple Boolean machine may be regarded as a discrete Markov process. Attention is directed here to the solution of this process by matrices of generating functions. Physical devices that may be used to analyze a Boolean system are discussed in Section 5, and a trinary counter is analyzed with respect to the theory of Sections 4 and 5 for purposes of illustration of the theory.

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SOME MATHEMATICAL REMARKS ON THE BOOLEAN MACHINE

Part I*

1. The Mathematical Boolean Function of a Real Independent Variable t and the Calculus of Such Functions

First let us define the general Boolean function F(t) of a real independent variable t.

Definition 1.1 F(t), where t lies in either a finite or an infinite interval, the range R of t, is said to be a general Boolean function or a GB-function if it has the following property.

For every value of the real variable, t in the range of definition. F(t) has one and only one of the abstract values, either 0 or I. That is,

$$F(t) = (\text{either } 0 \text{ or } I) \text{ for } t \in R. \tag{1.1}$$

Let us now consider the definition of the sum, product and negation of GB-functions.

Definition 1.2 If there are two functions, F(t) and G(t) defined as in Definition 1.1 over the same range R of t, then the sum, product and negation functions are, respectively,

$$F(t) + G(t), \; F(t)\, G(t) \text{ and } F'(t) \tag{1.2}$$

for each real value of t within R. The rules for negation, sum and product are given in Table 1.

Table 1

F(t)	G(t)	F'(t)	F(t) + G(t)	F(t) G(t)
0	0	I	0	0
0	I	I	I	0
I	0	0	I	0
I	I	0	I	I

*Author's Note: Part I of the Report "Some Mathematical Remarks on the Boolean Machine" is a corrected and revised copy of a report of the same part and title that was previously submitted to Computer Research Corporation on 1 August 1951.

Lemma 1.1 The sum, product and negation functions of GB-functions are GB-functions.

Proof: The lemma follows immediately from Table 1 and Definition 1.1.

Let us now define the left and right limits of a GB-function.

Definition 1.3 The left limit

$$F(t_o-) = \lim_{\epsilon \to 0} F(t_o - \epsilon) \tag{1.3}$$

of a GB-function F(t) is said to exist for $t = t_o$ if there exists $\delta > 0$ such that

$$F(t_o - \epsilon) = F(t - \delta)$$

for all ϵ where $\delta > \epsilon > 0$. If $F(t_o-)$ exists, then

$$F(t_o-) = F(t_o - \delta_1)$$

for any δ_1 where $\delta \geqslant \delta_1 > 0$. The condition for existence and the definition of a right limit are had if in the above the word "left" is replaced by "right" and minus signs are replaced by plus signs.

Definition 1.4 Suppose F(t) is a GB-function over the range R = (a, b) where $a \leqslant t \leqslant b$. If $F(t_o-)$ exists for $t_o \epsilon$ R and

$$F(t_o-) = F(t_o),$$

then F(t) is said to be left continuous at the point $t = t_o$. If $F(t_o+)$ exists and

$$F(t_o+) = F(t_o),$$

then F(t) is right continuous at the point $t = t_o$. If F(t) is both left and right continuous at the point $t = t_o$, then F(t) is continuous at the point $t = t_o$ or t_o is a point of continuity of the GB-function F(t). Suppose $F(t_o-)$ and $F(t_o+)$ exist at $t = t_o$, then the point t_o is said to be a point of simple discontinuity of the GB-function F(t) if t_o is not a point of continuity.

Definition 1.5 If the range R of t is finite, then F(t) is defined to be a B-function if it is a BG-function and every point t of R is a point of continuity except possibly for, at most, a finite set of points of simple discontinuity (including the end points of the range) as defined in Definition 1.4. If R is an infinite range, then $F(t)$ is a B-function if for every subinterval of R it is continuous everywhere except possibly for, at most, a finite set of points of simple discontinuity.

Definition 1.6 A B-function, as defined in Definition 1.5, is defined to be a B_r-function if $F(t)$ is right continuous for all values of t.

Definition 1.7 A B-function is defined to be a B_ℓ-function if $F(t)$ is left continuous for all values of t where t R.

Definition 1.8 A B-function $F(t)$ is a B_o-function if for every possible point $t = t_o$ of simple discontinuity, $F(t)$ is neither right nor left continuous at $t = t_o$.

Definition 1.9 Suppose $F(t)$ is a GB-function and that $t = t_o$ is a point of simple discontinuity of $F(t)$. Then t_o is called a right jump point of $F(t)$ if $F(t)$ is right continuous at $t = t_o$. If t_o is a right jump point and

$$F(t_o-) = 0, \text{ then } F(t_o) = I,$$

and $t = t_o$ is called a right up jump point of $F(t)$, otherwise t_o is a right down jump point. Left jump, left up or down jump, points are defined similarly. If $F(t)$ is neither left nor right continuous at $t = t_o$, then t_o is a spike point. If t_o is a spike point and

$$F(t_o-) = F(t_o+) = 0,$$

then $F(t_o) = I$ and $t = t_o$ is called an up spike point of $F(t)$, otherwise t_o is a down spike point.

From Definitions 1.6, 1.7, 1.8 and 1.9, the following theorem is clearly true:

Theorem 1.1 The only possible points of simple discontinuity of B_ℓ, B_r or B_o functions are, respectively, left jump points, right jump points or spike points.

The following theorem is clearly true by Lemma 1.1 and Definitions 1.3, 1.6, 1.7 and 1.8.

<u>Theorem 1.2</u> The sum, product and negation functions of B, B_ℓ, B_r or B_o functions are, respectively, B, B_ℓ, B_r or B_o functions.

From Theorem 1.2 and Definition 1.3 we clearly have the following lemma:

<u>Lemma 1.2</u> If F(t) and G(t) are B-functions, then

$$\lim_{\epsilon \to o} |F(t-\epsilon) + G(t-\epsilon)| = F(t-) + G(t-)$$

$$\lim_{\epsilon \to o} |F(t+\epsilon) + G(t+\epsilon)| = F(t+) + G(t+)$$

$$\lim_{\epsilon \to o} |F(t-\epsilon)\, F(t-\epsilon)| = F(t-)\, G(t-)$$

$$\lim_{\epsilon \to o} |F(t+\epsilon)\, G(t+\epsilon)| = F(t+)\, G(t+)$$

$$\lim_{\epsilon \to o} F'(t-\epsilon) = F'(t-)$$

and

$$\lim_{\epsilon \to o} F'(t+\epsilon) = F'(t+)$$

The next theorem will show that the left or right limit operation transforms a B-function of one type into a B-function of another type.

<u>Theorem 1.3</u> If F(t) is a B_r-function, then $G(t) = F(t-)$ is a B_ℓ-function. If F(t) is a B_ℓ function, then $G(t) = F(t+)$ is a B_r-function. If F(t) is a B_o function, then

$$F(t-) = F(t+) = I \text{ all } t \in R$$

if F(t) has down spikes, or

$$F(t-) = F(t+) = 0 \text{ all } t \in R$$

if F(t) has up spikes.

<u>Proof:</u> If F(t) is a B_r-function, suppose t_o is a point of simple discontinuity. By Definitions 1.4 and 1.3 there then exists $\delta > o$ such that

$$F(t_o + \delta) = F(t_o) = F(t_o + \epsilon)$$

and

$$F(t_o - \delta) = F(t_o - \epsilon)$$

all ϵ such that $o < \epsilon \leqslant \delta$. Thus for ϵ such that $\delta \geqslant \epsilon > o$ we have

$$G(t_o + \epsilon) = F[(t_o + \epsilon)-] = F(t_o + \epsilon) = F(t_o - \delta) = F(t_o)$$

Hence

$$G(t_o+) = F(t_o)$$

and $G(t)$ has a right limit at $t = t_o$, a right jump point of $F(t)$. Also for all ϵ, such that $o < \epsilon_1 \leqslant \delta/2 \leqslant \delta$ we have

$$G(t_o - \epsilon) = F[(t_o - \epsilon_1)-] = F(t_o - \epsilon_1) = F(t_o-) = G(t_o) \quad .$$

Therefore

$$G(t_o-) = G(t_o)$$

and $G(t)$ is left continuous at $t = t_o$. If t_o is a point of continuity of $F(t)$, then by an argument similar to the above,

$$G(t_o-) = G(t_o) = G(t_o+) \quad .$$

Hence t_o is also point of continuity of $G(t)$. We have satisfied the conditions for $G(t) = F(t-)$ to be a B_ℓ-function, thus the first part of the theorem is proved. The rest of the theorem follows by a similar argument.

If one thinks of the space of all B_ℓ-functions or all B_r-functions, then Theorem 1.3 establishes a one-to-one correspondence between the elements of the two spaces; moreover, Theorem 1.2 and Lemma 1.2 provide the machinery to show that this mapping is an algebraically isomorphic mapping. A further study of the various properties of these function spaces will be made at a later date (these function spaces form interesting topological rings).

Theorem 1.4 If $F(t)$ is a B_r-function, the transformations

$$d_{r-} F(t) = F(t-)\, F'(t)$$

$$d_{r+} F(t) = F'(t-)\, F(t) \qquad (1.4)$$

$$d_r F(t) = d_{r-} F(t) + d_{r+} F(t) = F(t-) \oplus F(t) \quad ,$$

where $\oplus$ is the Boolean ring sum, map $F(t)$ onto B_o-functions. The operators d_{r-}, d_{r+} and d_r map, respectively, only the down jump points of $F(t)$ into up spikes, only the up jump points of F(t) into up spikes and only the jump points of F(t) into up spikes. If F(t) is a B_ℓ-function, then the transformations

$$
\begin{aligned}
d_{\ell-}F(t) &= F(t)\,F'(t+) \\
d_{\ell+}F(t) &= F'(t-)\,F(t) \qquad\qquad (1.5) \\
d_\ell F(t) &= d_{\ell-}F(t) + d_{\ell+}F(t) = F(t) \oplus F(t+)
\end{aligned}
$$

map F(t) onto B_o-functions in a manner corresponding to the mappings (1.4).

<u>Proof:</u> By Theorem 1.3 F(t), F(t+) and F(t−) are B-functions. Hence from Theorem 1.2, $d_{r-}F(t)$, $d_{r+}F(t)$ and $d_rF(t)$ are B-functions and by Definition 1.2, Lemma 1.2 and Definition 1.9 the only points of discontinuity are spike points. Thus $d_{r-}F(t)$, $d_{r+}F(t)$ and $d_rF(t)$ are B_o-functions. The remainder of the theorem is clearly evident.

The Boolean ring plus sign $\oplus$, mentioned in the definition of $d_r F(t)$ and $d_\ell F(t)$ in Theorem 1.4, the product sign (·) and the sign + are related by the following rules:

$$
\begin{aligned}
F(t) + G(t) &= F(t) \oplus G(t) \oplus F(t)\,G(t) \\
F(t) \oplus G(t) &= F'(t)\,G(t) + F(t)\,G'(t) \qquad\qquad (1.6)
\end{aligned}
$$

and

$$F'(t) = I \oplus F(t) \quad .$$

where F(t) and G(t) are GB-functions. It can be shown that a Boolean algebra under the operations + and (·) is a Boolean ring under the operations $\oplus$ and (·). Under the operation $\oplus$, the Boolean ring is an additive group.

Henceforth, let us denote d_r, d_{r+}, d_{r-}, by d, d+, d− respectively, and let us deal with B_r-functions and B_o-functions unless otherwise specified. Moreover, let us denote B_r-functions by capital latin letters and B_o-functions by small Latin letters. By Theorem 1.3 for any result that is developed for B_r-functions there will be a corresponding result for B_ℓ-functions.

From Theorem 1.4 we see that d, d+ and d− are operators which map B_r-functions onto B_o-functions. Let us now consider the inverse images of these mappings. Suppose first we are given the operator equation

$$dX(t) = p(t) \tag{1.7}$$

where p(t) is a B_o-function with up spike points at only the points t_o, $t_{\pm 1}$, $t_{\pm 2}$, ... $t_{\pm i}$, ... We wish to know all the B_r-functions X(t) which satisfy (1.7) if any; those functions X(t) which satisfy (1.7) are solutions of the operator equation (1.7) or in this particular case, what might be called integrals of ρ(t). From Theorem 1.4, two solutions that satisfy (1.7) are

$$X(t) = S(t) \quad \text{or} \quad I \oplus S(t) \tag{1.8}$$

where S(t) has only down jump points for $t = t_o$, $t_{\pm 2}$, ... $t_{\pm 2i}$, ... and only up jump points for $t = t_{\pm 1}$, $t_{\pm 3}$, ... $t_{\pm 2i+1}$, ... In order to show that the solutions, given by (1.8), are the only B_r-functions which satisfy (1.7) let us consider the following lemma:

<u>Lemma 1.3</u> The only B_r- and B_o-functions which satisfy the operator equation

$$dX(t) = 0 \qquad \text{all } t\epsilon R \tag{1.9}$$

are

$$X(t) = 0 \qquad \text{all } t\epsilon R$$

or

$$I \qquad \text{all } t\epsilon R$$

<u>Proof:</u> By Theorem 1.4, Equation (1.9) may be written as

$$X(t) \oplus X(t-) = 0 \qquad \text{all } t\epsilon R$$

or

$$X(t) = X(t-) \qquad \text{all } t\epsilon R$$

But this means that X(t) is a left continuous function. But by hypothesis if X(t) is a B_r-function, it is right continuous. Thus X(t) is continuous at every point $t\epsilon R$. Suppose $X(t) \neq 0$ all $t\epsilon R$ and $X(t) \neq I$ all $t\epsilon R$, then by Theorem 1.1, there must exist a right jump point at some point $t_o\epsilon R$, which is a contradiction to the continuity of $X(t)$. If $X(t)$ is a B_o-function, then by Theorem 1.3

$$X(t) = X(t-) = I \qquad \text{all } t \epsilon R$$

or

$$= 0 \qquad \text{all } t \epsilon R$$

Hence Lemma is proved.

If we had not restricted Lemma 1.3 to B_r and B_o functions, we would have found that every B_ℓ-function satisfies (1.9). This is true since every B_ℓ-function $X(t)$ is left continuous, thereby satisfying (1.9) identically.

Theorem 1.5 The only B_r-functions which satisfy the operator equation are those given by (1.8). The only B_o-functions that satisfy (1.7) are

$$X(t) = p(t)$$

and

$$X(t) = I \oplus p(t) \quad .$$

If $p(t)$ in (1.7) has at least one up spike point, then there are no B_ℓ-functions which ~~satisfy (1.7).~~

Proof: Suppose there exists a B_r-function $F(t)$ other than $X(t) = S(t)$ or $I \oplus S(t)$ which satisfies (1.7). Then

$$X(t) = p(t) \qquad \text{and} \qquad dF(t) = p(t)$$

or

$$dX(t) \oplus dF(t) = p(t) \oplus p(t) = 0 \qquad \text{all } t \epsilon R \quad .$$

Thus

$$\begin{aligned} 0 &= [X(t) \oplus X(t-)] \oplus [F(t) \oplus F(t-)] \\ &= [F(t) \oplus X(t)] \oplus [F(t-) \oplus X(t-)] \\ &= [F(t) \oplus X(t)] \oplus \lim_{\epsilon \to 0} [F(t-\epsilon) + X(t-\epsilon)] \end{aligned}$$

by Lemma 1.2. Therefore by (1.4) we have

$$d[F(t) \oplus X(t)] = 0 \qquad \text{all } t \epsilon R,$$

which by Lemma 1.3 and Theorem 1.2 implies

$$F(t) \oplus X(t) = 0 \qquad \text{all } t \epsilon R$$

or

$$I \qquad \text{all } t \epsilon R \quad .$$

Hence

$$F(t) = X(t) \qquad \text{or} \qquad I \oplus X(t) \qquad \text{all } t \in R \quad ,$$

which verifies the uniqueness of the B_r-function solutions, given by (1.8). The third sentence of the theorem is evident from the sentence preceeding the theorem. The second statement of the theorem follows from the evident identities,

$$dp(t) = p(t)$$

and

$$d\left(I \oplus p(t)\right) = p(t)$$

If $p(t) = 0$ for $t < 0$ and if for $t \geq 0$, $p(t)$ has up spike points, then the operator equation (1.7) is somewhat analogous to a simple ordinary first order differential equation

$$\frac{dY(t)}{dt} = a(t)$$

where $T(o)$ is the initial condition. If $X(0-)$ is given as an initial condition, then by Theorem 1.5, (1.7) has a unique solution $S(t)$ for $t \geq 0$, and there are only two possible solutions which are B_r-functions. Let us pursue this analogy further in the following theorem.

<u>Theorem 1.6</u> Suppose $p(t)$ and $q(t)$ are B_o-functions such that for $t < 0$, $p(t) = q(t) = 0$, and for $t \geq 0$ that $p(t)$ and $q(t)$ possess spike points. Then the operator equation

$$dX(t) \oplus p(t)\, X(t-) = q(t) \quad , \tag{1.10}$$

where initially $X(0-)$ is either 0 or I, has a unique solution $S(t)$ for $t \geq 0$ which is a B_r-function. The equation has two and only two possible B_r-function solutions, depending entirely on the value of $X(0-)$.

<u>Proof:</u> Let us construct a B_r-function $S(t)$ which satisfies (1.10). Let

$$S(t) = X(0-) \quad \text{for} \quad t < 0,$$

and for $t \geq 0$ let $S(t)$ be defined by Table 2.

Table 2

p(t)	q(t)	S(t–)	S(t)
0	0	0	0
0	0	I	I
0	I	0	I
0	I	I	0
I	0	0	0
I	0	I	0
[illegible]	I	0	I
I	I	I	I

It is clear from the definition of the B_r-function S(t) that S(t) has jump points at a point t only when t is not a spike point of p(t), t is a spike point of q(t) and S(t–) = 0, or t is a spike point of p(t), t is not a spike point of q(t) and S(t–) = I or t is a spike point of both p(t) and q(t) and S(t–) = 0. It is further evident that the B_r-function S(t) satisfies (1.10) identically.

Let us consider the uniqueness of the solution to (1.10) that was constructed above. Suppose F(t), a B_r-function, satisfies (1.10) as well as S(t), where for $t < 0$, F(t) is defined by $F(t) = X(0-)$, the initial condition. Thus we have

$$dF(t) \oplus p(t)\, F(t-) = q(t)$$

and

$$dS(t) \oplus p(t)\, S(t-) = q(t)$$

for all $t \in R$. Adding these two expressions, we obtain

$$d\left[F(t) \oplus S(t)\right] + p(t)\left[F(t-) \oplus S(t-)\right] = 0 \tag{1.11}$$

for all $t \in R$ and where $F(t) = S(t)$ for $t < 0$. If we show that $Y(t) = 0$ for all $t \in R$ is the only B_r-function solution to the operator equation

$$dY(t) \oplus p(t)\, Y(t-) = 0 \tag{1.12}$$

where initially $Y(t) = 0$ for $t < 0$, then by (1.11) $F(t) = S(t)$ for all $t \in R$ and the theorem is proved. Suppose the contrary, that $Y(t) \neq 0$ for all $t \in R$, then there exists a point t_o of simple discontinuity such that for all $t < t_o$, $Y(t) = 0$. The point t_o must be an up jump point since the solution is to be a B_r-function. Thus

$$Y(t_o-) = 0$$

and

$$Y(t_o) = I \quad .$$

Substituting these values into the left of (1.12) we have

$$dY(t_o) + p(t_o)\, Y(t_o-) = I$$

which is a contradiction. Hence theorem is proved.

If in the operator equation (1.10) we let

$$b(t) = q(t)$$

and

$${}_ob(t) = b(t) + p(t),$$

then (1.10) becomes

$$dX(t) \oplus [b(t) \oplus {}_ob(t)]\, X(t-) = b(t)$$

or

$$X(t) = b(t) \oplus [I \oplus b(t) \oplus {}_ob(t)]\, X(t-) \quad . \tag{1.13}$$

From (1.6) we may rewrite (1.13) as

$$X(t) = b(t)\, X'(t-) + {}_ob'(t)\, X(t-) \quad . \tag{1.14}$$

Equation (1.10) and its equivalent (1.14) may be analysed approximately by a physical device called a two input flip flop (for example, the Ecclos-Jordan flip flop). The flip flop is a bistable state devise which is triggered by pulses, the physical approximation to a spike. In this sense Equation (1.14) may be termed the ideal flip-flop equation with the two B_o function inputs $b(t)$ and ${}_ob(t)$.

Let us now define a special class of B_r functions, called the $B_\tau(\alpha)$ functions.

Definition 1.7 A $B_\tau(\alpha)$ function is a B_r-function $F(t)$ over the infinite range $\infty < t < \infty$ which has jump points only at the points

$$\alpha,\ \alpha\pm\tau,\ \alpha\pm2\tau,\ \ldots,\ \alpha\pm n\tau,\ \ldots$$

$(n = 0, 1, 2, 3, \ldots)$. The constant α is called the phase of $F(t)$ and τ is the translation period of $F(t)$.

From Definition 1.7 the following lemma is evident.

Lemma 1.4 If $0 \leq \beta < \tau$, then the $B_\tau(\beta)$-function $F(t)$ is a $B_\tau(\beta + n\tau)$-function for $n = 0, \pm 1, \pm 2, \ldots$ and conversely. β is called the residual phase of $F(t)$.

Let us consider the two $B_{\tau/2}(0)$ functions $\alpha(t)$ and ${}_o\alpha(t)$ which have the special properties:

$$\alpha(t) = {}_o\alpha(t) = 0 \text{ for } t_n = (2n)\tau/2 < t \leq (2n+1)\,\tau/2 \tag{1.15}$$

$(n = 0, 1, 2, \ldots)$,

$$\left.\begin{array}{l} \alpha(t) = 0 \text{ or } I \\ {}_o\alpha(t) = 0 \text{ or } I \end{array}\right\} \text{ for } (2n+1)\,\tau/2 \leq t < (2n+2)\,\tau/2 = t_{n+1}$$

$(n = 0, 1, 2, \ldots)$,

and

$$\left.\begin{array}{l} \alpha(t) = 0 \text{ or } I \\ {}_o\alpha(t) = 0 \text{ or } I \end{array}\right\} \text{ for } -\infty < t < 0 \quad .$$

Now consider the operator equation

$$X(t) = d_-[\alpha(t)]\; X'(t-) + d'_-[{}_o\alpha(t)]\; X(t-) \tag{1.16}$$

where initially $X(0-) = S(0-)$. (1.16) is the ideal flip flop equation (1.14) where one input consists of up spikes occurring at the down jump points of $\alpha(t)$ and the other input consists of up spikes occurring at the down jump points of ${}_o\alpha(t)$.

Lemma 1.5 The unique solution $S(t)$ of (1.16) is a $B_\tau(0)$-function which as well is the unique solution of the Boolean difference equation

$$Y(t) = \alpha(t-\tau/2)\alpha'(t)Y'(t-\tau/2) + [{}_o\alpha(t-\tau/2)\,{}_o\alpha(t-\tau/2)]'\,Y(t-\tau/2) \tag{1.17}$$

for $t \geq \tau/2$ where the initial condition of (1.17) is $Y(t) = S(t)$ for $0 \leq t < \tau/2$.

Proof: That the solution of (1.16) is a B(0)-function is evident from (1.15) and Theorem 1.6. To show that S(t) satisfies (1.17) consider the following two cases:

Case I: $t_i \leq t < t_i + \tau/2$ $(i = 1, 2, 3, \ldots)$

For this case $\alpha(t) = {}_o\alpha(t) = 0$ and from (1.17)

$$Y(t) = \alpha(t - \tau/2)\, Y'(t - \tau/2) + {}_{o}\alpha'(t - \tau/2)\, Y(t - \tau/2) \quad . \tag{1.18}$$

But also for this range of t

$$\alpha(t - \tau/2) = \alpha(t_i -) \quad ,$$

$${}_{o}\alpha(t - \tau/2) = {}_{o}\alpha(t_i -) \quad ,$$

$$S(t - \tau/2) = S(t_i -) \quad ,$$

$$\alpha(t_i) = {}_{o}\alpha(t) = 0$$

and also from (1.16)

$$S(t_i) = \alpha(t_i -)\, S'(t_i -) + {}_{o}\alpha'(t_i -)\, S(t_i -) \quad .$$

Hence

$$S(t) = \alpha(t - \tau/2)\, S'(t - \tau/2) + {}_{o}\alpha'(t - \tau/2)\, S(t - \tau/2) \quad . \tag{1.19}$$

From (1.19) and (1.18) we see that $S(t)$ satisfies (1.17) when $t_i \leqslant t < t_i + \tau/2$.

Case II: $\quad t_i + \tau/2 \leqslant t < t_{i+1} \qquad (i = 0, 1, 2, \ldots)$

For this case

$$\alpha(t - \tau/2) = 0$$

and

$${}_{o}\alpha(t - \tau/2) = 0 \quad ,$$

thus

$$Y(t) = Y(t - \tau/2) \tag{1.20}$$

But since $S(t)$ is a $B_\tau(0)$-function

$$S(t) = S(t - \tau/2) \quad . \tag{1.21}$$

From (1.20) and (1.21) we see that $S(t)$ satisfies (1.17) when $t_i + \tau/2 \leqslant t < t_{i+1}$. From cases I and II we see that $S(t)$, the solution of (1.16) is a solution of (1.17) for $t \geqslant \tau/2$. The uniqueness of the solution may be established in a manner similar to the uniqueness-proof made in the next theorem.

Let us now define the so-called clock function $E(t)$.

Definition 1.8 The clock function $E(t)$ is a $B_{\tau/2}(0)$-function which has the following properties:

$$E(t) = I \quad \text{for} \quad -\infty < t < 0$$
$$= 0 \quad \text{for} \quad t_i = (2i)\,\tau/2 \leqslant t < (2i+1)\,\tau/2$$
$$= I \quad \text{for} \quad (2i+1)\,\tau/2 \leqslant t < (2i+2)\,\tau/2 = t_i + 1 \quad .$$

Now consider the $B_\tau(0)$ functions $\beta(t)$ and ${}_o\beta(t)$ with the properties:

$$\left.\begin{array}{l} \beta(t) = 0 \text{ or } I \\ {}_o\beta(t) = 0 \text{ or } I \end{array}\right\} \text{ for all } t, -\infty < t < 0 \qquad (1.22)$$

and

$$\left.\begin{array}{l} \beta(t) = 0 \text{ or } I \\ {}_o\beta(t) = 0 \text{ or } I \end{array}\right\} \text{ for } t_i = i\tau \leqslant t < (i+1)\tau = t_{i+1}$$

where $i = 0, 1, 2, \ldots$ From Definition 1.8 and (1.22) it is clearly evident that $E(t)\beta(t)$ and $E(t)_o\beta(t)$ have the same properties as $a(t)$ and ${}_oa(t)$, given by (1.15). Moreover, by Lemma 1.6 the solution $S(t)$ of

$$X(t) = d_-[E(t)\,\beta(t)]\; X'(t-) + d'_-[E(t)_o\beta(t)]\; X(t-) \quad , \qquad (1.23)$$

where the initial condition is $X(0-)$, is also the solution of

$$Y(t) = E(t-\tau/2)\,\beta(t-\tau/2)\,[E(t)\,\beta(t)]'\; Y'(t-\tau/2)$$
$$+ [E(t-\tau/2)\;{}_o\beta(t-\tau/2)\,\{E(t)\;{}_o\beta(t)\}]'\; Y(t-\tau/2) \qquad (1.24)$$

for $t \geqslant \tau/2$ where initially $Y(t) = S(t)$ when $0 \leqslant t < \tau/2$.

Theorem 1.7 The $B_\tau(0)$-function $S(t)$ which is the unique solution of (1.23) and (1.24) is also the unique solution of the Boolean difference equation

$$Z(t) = \beta(t-\tau)\,Z'(t-\tau) + {}_o\beta'(t-\tau)\,Z(t-\tau) \qquad (1.25)$$

for $t \geqslant \tau$ where initially $Z(t) = S(t)$ when $0 \leqslant t < \tau$ and $\beta(t)$, ${}_o\beta(t)$ are $B_\tau(0)$ functions, defined by (1.22).

Proof: From Definition 1.8 we have

$$E(t) = E'(t-\tau/2) \quad \text{when } t \geqslant 0. \qquad (1.26)$$

Now from (1.24) when $t \geqslant \tau$,

$$S(t) = \beta(t-\tau/2)\, E'(t)\, S'(t-\tau/2) + \left[{}_o\beta(t-\tau/2)\, E'(t)\right]'\, S(t-\tau/2)$$

$$= \beta(t-\tau/2)\, E'(t)\left\{\left[\beta(t-\tau)\, E'(t-\tau/2)\right]'\, S'(t-\tau) + {}_o\beta(t-\tau/2)\, E'(t-\tau/2)\, S(t-\tau)\right\}$$

$$+ \left[{}_o\beta(t-\tau/2)\, E'(t)\right]'\left\{\beta(t-\tau)\, E'(t-\tau/2)\, S'(t-\tau) + \left[{}_o\beta(t-\tau)\, E'(t-\tau/2)\right]'\, S(t-\tau)\right\}.$$

If we expand and use (1.26) this becomes

$$S(t) = \beta(t-\tau/2)\, E'(t)\, S'(t-\tau) + \beta(t-\tau)\, E'(t-\tau/2)\, S'(t-\tau)$$

$$+ {}_o\beta'(t-\tau/2)\; {}_o\beta'(t-\tau)\, S(t-\tau) + {}_o\beta'(t-\tau)\, E(t)\, S(t-\tau)$$

$$+ {}_o\beta'(t-\tau/2)\, E(t-\tau/2)\, S(t-\tau) \quad . \tag{1.27}$$

Consider the following two regions:

<u>Region I:</u> $t_i \leq t < t_i + \tau/2$, $(i = 1, 2, 3, \ldots)$

In this region

$$E(t) = 0 \; ,$$

$$E(t-\tau) = I \; ,$$

$$\beta(t-\tau/2) = \beta(t-\tau)$$

and

$${}_o\beta(t-\tau/2) = \beta(t-\tau) \; .$$

Thus by (1.27)

$$S(t) = \beta(t-\tau/2)\, S'(t-\tau) + {}_o\beta'(t-\tau/2)\, S(t-\tau)$$

$$= \beta(t-\tau)\, S'(t-\tau) + {}_o\beta'(t-\tau)\, S(t-\tau) \quad .$$

Hence in this region $S(t)$ satisfies (1.25).

<u>Region II:</u> $t_i + \tau/2 \leq t < t_{i+1}$ $(i = 1, 2, 3, \ldots)$

In this region $E(t) = I$ and $E(t-\tau) = 0$. Thus by (1.27)

$$S(t) = \beta(-\tau)\, S'(t-\tau) + {}_o\beta'(t-\tau)\, S(t-\tau) + {}_o\beta'(t-\tau)\; {}_o\beta'(t-\tau)\, S(t-\tau)$$

$$= \beta(t-\tau)\, S'(t-\tau) + {}_o\beta'(t-\tau)\, S(t-\tau)\, \left[I + {}_o\beta'(t-\tau)\right]$$

$$= \beta(t-\tau)\, S'(t-\tau) + {}_o\beta'(t-\tau)\, S(t-\tau) \quad .$$

Hence in this region $S(t)$ satisfies (1.25). Since $S(t)$ satisfies (1.25) in Regions I and II, it satisfies (1.25) for $t \geq \tau$. For the uniqueness of the solution $S(t)$ note by (1.6) that (1.25) may be rewritten as

$$Z(t+\tau) = \beta(t) \oplus \left[\beta(t) \oplus {}_o\beta(t)\right]\; Z(t) \oplus Z(t) \tag{1.28}$$

for $t \geq 0$. Suppose there is another solution $T(t)$ with initial condition $T(t) = S(t)$ for $0 \leq t < \tau$ which also satisfies (1.28). Also let $L(t) = S(t) + T(t)$, then by (1.28) we have the difference equation

$$L(t + \tau) = [\beta(t) \oplus {}_o\beta(t) \oplus I] \; L(t) \tag{1.29}$$

with initial condition $L(t) = 0$ for $0 \leq t < \tau$. Consider any arbitrary point t_o where $0 \leq t_o < \tau$. Then by (1.29) $L(t_o + \tau) = 0$. Suppose for the purposes of induction that $L(t_o + n\tau) = 0$, then we have by (1.29) $L(t_o + (n+1)\tau) = 0$. Hence the induction is complete and $L(t_o + n\tau) = 0$ for all n ($n = 0, 1, 2, \ldots$). Since t_o was arbitrary for $0 \leq t_o < \tau$, we thus have $L(t) = 0$ for all t, $t \geq 0$. Hence $S(t) = T(t)$ for all t, $t \geq 0$, and the solution $S(t)$ is unique. Hence theorem is proved.

Equation (1.25) is called the Boolean difference equation of the clocked flip ~~flop. Since (1.25) is independent of E(t), the clock function, the physical flip flop~~ which analyzes this equation may be considered as a box with two input and two ~~output ($Z_i'(t)$ is ordinarily available as an output from the physical flip flop) functions~~ all of the same nature, $B_\tau(0)$ functions. Since $B_\tau(0)$ functions $F(t)$ have the property

$$F(t) = F(t_i) \qquad \text{for } t_i \leq t < t_i + \tau \tag{1.30}$$

where t_i is a possible jump point, (1.25) may be considered entirely in terms of its jump points as

$$Z(t_i) = \beta(t_{i-1}) \, Z'(t_{i-1}) + {}_o\beta'(t_{i-1}) \, Z(t_{i-1}) \;, \quad (i = 1, 2, 3, \ldots) \;, \tag{1.31}$$

where initially $Z(t_o) = S(t_o)$ (either 0 or I). Equation (1.31) may also be called the state equation of the clocked flip flop since by (1.30) the value or state of $F(t)$ is completely determined by $F(t_i)$ for an interval of at least τ where t_i is a possible jump point. If we let

$$Z_i = Z(t_i), \; \beta(t_i) = \beta_i \text{ and } {}_o\beta(t_i) = {}_o\beta_i \tag{1.32}$$

for $i = 0, 1, 2, \ldots$, then (1.31) becomes

$$Z_i = \beta_{i-1} \, Z_{i-1}' + {}_o\beta_{i-1}' \, Z_{i-1} \qquad (i = 1, 2, \ldots) \tag{1.33}$$

where initially Z_o = (either 0 or I). Equation (1.33) is the state equation of the clocked flip flop where the dependence on t has been removed; (1.33) is a discreet Boolean difference equation. Let us now consider a general system of Boolean difference equations which reduce to a set of equations of the type of (1.33).

Let $Z_i^{(j)}$ $(j = 1, \ldots N)$ be a set of dependent functions defined as in (1.32). Let $\omega_i^{(k)}$ $(k = 1, \ldots M)$ be a set of independent functions, given for $i = 0, 1, 2, \ldots$. Now consider the system of Boolean difference equations

$$Z_{i+1}^{(j)} = f^{(j)} (Z_i^{(1)}, Z_i^{(2)}, \ldots Z_i^{(N)}; \omega_i^{(1)}, \omega_i^{(2)}, \ldots \omega_i^{(M)}) \qquad (1.34)$$

where $(j = 1, \ldots N)$, $(i = 0, 1, 2, \ldots)$ and initially $Z_o^{(j)}$ = (either 0 or I) for $j = 1, \ldots N$. By de Morgan's theorem (see Section 2) (1.34) may be written as

$$Z_{i+1}^{(j)} = \beta^{(j)} (Z_i^{(1)}, \ldots Z_i^{(j-1)}, Z_i^{(j+1)}, \ldots Z_i^{(N)}; \omega_i^{(1)}, \ldots \omega_i^{(M)}) Z_i'$$
$$+ {}_o\beta^{(j)'} (Z_i^{(1)}, \ldots Z_i^{(j-1)}, Z_i^{(j+1)}, \ldots Z_i^{(N)}; \omega_i^{(1)}, \ldots \omega_i^{(M)}) Z_i \qquad (1.35)$$

where $(j = 1, \ldots N)$, $(i = 0, 1, 2, \ldots)$, initially $Z_o^{(j)}$ = (either 0 or I) for $j = 1, \ldots N$ and both $\beta_{i-1}^{(j)}$ and ${}_o\beta_{i-1}^{(j)}$ are independent of $Z_i^{(j)}$ and are in terms of sums and products of the remaining variables $Z_i^{(j)}$, $Z_i^{(j)'}$, $\omega_i^{(k)}$ and $\omega_i^{(k)'}$ for $(j = 1, \ldots N)$ and $(k = 1, \ldots M)$. Equations (1.33) and (1.35) show that the simple Boolean system, given in (1.34) may be analysed physically by a machine consisting of N clocked flip flops for the dependent variables and suitable physical devices for producing the sum and product of the various variables. Such a machine will be called the simple Boolean machine.

The best examples of simple Boolean machines known to this author are the Maddidas and (or) universal computers being built or considered by Computer Research Corporation, Northrop Aircraft Inc., Hughes Aircraft, Cal. Tech., and others. It is this author's belief that all the electronic and relay digital computers in existence today may be interpreted as simple Boolean machines if the various elements of these machines are regarded in an appropriate manner, but this has yet to be proved.

There are more advanced Boolean systems of difference equations (many of which can be shown to reduce to the simple Boolean machine) which could be analyzed physically by what we will denote generally by Boolean machines. The study of more advanced systems will be our goal in part for the future as well as the study of the inherent physical problems of the Boolean machine.

By stating that a Boolean machine is the analyzer of a Boolean system (a mathematical system), we are saying that the processes of the machine and the system are analogues of one another. Therefore we may say that a Boolean machine is an analogue to a Boolean system and vice versa. This statement achieves an important conceptual step in the subject of digital computing machines. The Boolean system when regarded as an idealized mathematical model of a Boolean machine should do much to unify the different design techniques and physical devices involved in the many different digital computing machines that have and will be considered. The different Boolean systems which can be physically realized afford a hierarchy of new Boolean machines. When the physical principles of a Boolean machine are considered in conjunction with the mathematical principles of its model, the Boolean system, a general physical theory of the Boolean machine should be the ultimate result.

In concluding this section, let us consider some examples of physical approximations to $B_T(0)$ functions. Approximations to such functions occur as a sequence of high or low voltages or currents, a sequence of magnetized or unmagnetized spots on a magnetizable surface, a sequence of white or black, punched or unpunched spots on paper or other material, a sequence of charged or uncharged spots on the surface of a dielectric, as a sequence of two different materials stacked one upon the other, and so forth. The physical devices in a physical system which produce, which are acted upon or algebraically combine, both statically and dynamically, the physical $B_T(0)$ functions make up the Boolean machine. In the next section, we will introduce probability theory to the Boolean system.

2. The Introduction of Probability Theory into the Boolean System

Let us suppose we have n Boolean variables $\alpha_1, \ldots \alpha_n$ where each may have either the value 0 or 1. Now consider any function

$$x = x(\alpha_1, \alpha_2, \ldots \alpha_n) \qquad (2.1)$$

of $\alpha_1, \alpha_2, \ldots \alpha_n$ which has only the values 0 or 1. Let

$$|x| = \left\{(\alpha_1, \alpha_2, \ldots \alpha_n) : x(\alpha_1, \alpha_2, \ldots \alpha_n) = I\right\} \tag{2.2}$$

be the set of n-tuples $(\alpha_1, \alpha_2, \ldots \alpha_n)$ such that $x(\alpha_1, \alpha_2, \ldots \alpha_n) = I$.

<u>Lemma 2.1</u> If x and y are functions of $\alpha_1, \ldots \alpha_n$, as given by (2.1), then

$$|x| \,\Omega\, |y| = |xy| \tag{2.3}$$

where Ω is the set intersection operation and the bars are defined by (2.2)

<u>Proof:</u> By (2.2) we have

$$xy = \left\{(\alpha_1, \alpha_2, \ldots \alpha_n) : x(\alpha_1, \ldots \alpha_n)\, y(\alpha_1, \ldots \alpha_n) = I\right\}$$

and

$$|x| \,\Omega\, |y| = \left\{(\alpha_1, \alpha_2, \ldots \alpha_n) : x(\alpha_1, \ldots \alpha_n) = I\right\} \Omega \left\{(\beta_1, \ldots \beta_n) : y(\beta_1, \ldots \beta_n) = I\right\}.$$

Now suppose the n-tuple $(\gamma_1, \ldots \gamma_n)$ is contained in $x \,\Omega\, y$ or

$$(\gamma_1, \ldots \gamma_n) \quad |x| \,\Omega\, |y|$$

then

$$x(\gamma_1, \ldots \gamma_n) = I = y(\gamma_1, \ldots \gamma_n)$$

and hence

$$x(\gamma_1, \ldots \gamma_n)\, y(\gamma_1, \ldots \gamma_n) = I$$

which implies

$$|x| \,\Omega\, |y| \subset |xy| \tag{2.4}$$

where $\subset$ stands for set inclusion. On the other hand, suppose

$$(\delta_1, \ldots \delta_n) \in |xy|$$

then

$$x(\delta_1, \ldots \delta_n)\, y(\delta_1, \ldots \delta_n) = I$$

and by Table 3 we have

Table 3

x	y	xy	x + y
0	0	0	0
0	I	0	I
I	0	0	I
I	I	I	I

$$x(\delta_1, \dots \delta_n) = I \quad \text{and} \quad y(\delta_1, \dots \delta_n) = I$$

which implies

$$(\delta_1, \dots \delta_n) \in |x| \cap |y|$$

which in turn implies

$$|xy| \subset |x| \cap |y| \tag{2.5}$$

Combining (2.4) and (2.5) we see that (2.3) is true. Hence lemma is proved.

Lemma 2.2 If x and y are functions of $\alpha_1, \dots \alpha_n$, as given by (2.1), then

$$|x| \cup |y| = |x + y| \tag{2.6}$$

where $\cup$ is the set union operation and the bars are defined by (2.2).

Proof: By (2.2) we have

$$|\bar{x}| \cup |\bar{y}| = \left\{(\alpha_1, \dots \alpha_n) : \bar{x}(\alpha_1, \dots \alpha_n) = \bar{I}\right\} \cup \left\{(\beta_1, \dots \beta_n) : \bar{y}(\beta_1, \dots \beta_n) = \bar{I}\right\}$$

and

$$|x + \bar{y}| = \left\{(\alpha_1, \dots \alpha_n) : x(\alpha_1, \dots \alpha_n) + y(\alpha_1, \dots \alpha_n) = \bar{I}\right\}$$

Suppose for the n-tuple $(\gamma_1, \dots \gamma_n)$ we have

$$(\gamma_1, \dots \gamma_n) \in |x + y|$$

then

$$x(\gamma_1, \dots \gamma_n) + y(\gamma_1, \dots \gamma_n) = I,$$

which by Table 3 obtains:

$$x(\gamma_1, \dots \gamma_n) = I \quad \text{and} \quad y(\gamma_1, \dots \gamma_n) = 0 \tag{A}$$

which implies

$$(\gamma_1, \dots \gamma_n) \in |x| \subset |x| \cup |y|$$

or

$$x(\gamma_1, \dots \gamma_n) = 0 \quad \text{and} \quad y(\gamma_1, \dots \gamma_n) = I \tag{B}$$

which implies

$$(\gamma_1, \dots \gamma_n) \in |y| \subset |x| \cup |y|$$

or

$$x(\gamma_1, \dots \gamma_n) = I \quad \text{and} \quad y(\gamma_1, \dots \gamma_n) = I \tag{C}$$

which implies

$$(\gamma_1, \dots \gamma_n) \in |x| \cup |y|$$

By (A), (B) and (C) we thus have

$$|x + y| \subset |x| \cup |y| \quad . \tag{2.7}$$

Now suppose that

$$(\delta_1, \ldots \delta_n) \in |x| \cup |y|$$

then

$$x(\delta_1, \ldots \delta_n) = I \quad \text{and} \quad y(\delta_1, \ldots \delta_n) = 0 \tag{D}$$

or

$$x(\delta_1, \ldots \delta_n) = 0 \quad \text{and} \quad y(\delta_1, \ldots \delta_n) = I \tag{E}$$

or

$$x(\delta_1, \ldots \delta_n) = I \quad \text{and} \quad y(\delta_1, \ldots \delta_n) = I \quad . \tag{F}$$

By Table 3 we have

$$x(\delta_1, \ldots \delta_n) + y(\delta_1, \ldots \delta_n) = I$$

for (D), (E) and (F) which implies

$$(\delta_1, \ldots \delta_n) \in |x + y|$$

or

$$|x| \cup |y| \subset |x + y| \quad . \tag{2.8}$$

If we combine (2.7) and (2.8), we see that (2.6) is true. Hence lemma is proved.

Let us call the total set of n-tuples, $(\alpha_1, \ldots \alpha_n)$, Ω . Then

$$\Omega = \left\{(\alpha_1, \ldots \alpha_n)\right\} = \left\{(0, \ldots 0, 0),\ (0, \ldots 0, I), \ldots, (I, \ldots I, I)\right\} \tag{2.9}$$

The function of the type, given by (2.1), which is I for every element of Ω is I, thus

$$|I| = \Omega \ . \tag{2.10}$$

Let us consider the complementary set of $|x|$ which we will denote by $\overline{|x|}$. By (2.9) we have

$$\begin{aligned}\overline{|x|} &= \Omega - |x| = \Omega - \left\{(\alpha_1, \ldots \alpha_n) : x(\alpha_1, \ldots \alpha_n) = I\right\} \\ &= \left\{(\beta_1, \ldots \beta_n) : x(\beta_1, \ldots \beta_n) = 0\right\} = \left\{(\beta_1, \ldots \beta_n) : x'(\beta_1, \ldots \beta_n) = I\right\} \\ &= |x'| \quad .\end{aligned}$$

Thus

$$\overline{|x|} = |x'| \quad . \tag{2.11}$$

From (2.10) and (2.11) we have

$$|0| = 0 \tag{2.12}$$

where the zero on the right designates the empty set.

By de Morgan's theorem,

$$\begin{aligned} x(\alpha_1, \ldots \alpha_n) &= x(0, \ldots 0, 0)\, \alpha'_1 \ldots \alpha'_{n-1}\, \alpha'_n \\ &+ x(0, \ldots 0, 0)\, \alpha'_1 \ldots \alpha'_{n-1}\, \alpha_n \\ &+ \ldots + x(I, \ldots I, I)\, \alpha_1 \alpha_2, \ldots \alpha_{n-1}\, \alpha_n \quad , \end{aligned}$$

we see that there is a one-to-one correspondence between every subset of Ω and every Boolean function x of the n-Boolean variables $\alpha_1, \ldots \alpha_n$. Or specifically

$$x \longleftrightarrow |x| \tag{2.13}$$

as defined by (2.2). From Lemma 2.1, Lemma 2.2, (2.10), (2.11), (2.12) and (2.13) we have the theorem:

Theorem 2.1 The algebra of the 2^{2n}-Boolean functions of the n-Boolean variables $\alpha_1, \ldots \alpha_n$ is algebraically isomorphic to the algebra $\mathfrak{F}$ of all subsets of $\Omega = \{(\alpha_1, \ldots \alpha_n)\}$.

We may now form a field of probability if we allow three postulates to the properties of $\mathfrak{F}$, call the n-tuples $(\alpha_1, \ldots \alpha_n)$ elementary events and call the elements of $\mathfrak{F}$, $|x|$, random events. The postulates are (1):

I. To each set $|x|$ of $\mathfrak{F}$ there is assigned a non-negative real number $P(|x|)$. The number $P(|x|)$ is called the probability of the event $|x|$ or $x = I$.

II. $P(\Omega) = P(|I|) = 1$.

III. If $|x| \cap |y| = |xy| = |0| = 0$, then
$$P(|x| \cup |y|) = P(|x + y|) = P(|x|) + P(|y|) \quad .$$

(1) A. N. Kolmogorov, Theory of Probability, Chelsea (1950).

From I, II and III and (2.3), (2.6) and (2.11) it is not difficult to show

$$P(|x'|) = 1 - P(|x|)$$
$$P(|x+y|) = P(|x|) + P(|y|) - P(|xy|) \qquad (2.14)$$
$$P(|xy|) = P_{|x|}(|y|)\, P(|x|) = P_{|y|}(|x|)\, P(|y|)$$

where $P_{|x|}(|y|)$ is the conditional probability of the event $|y|$ given the event $|x|$. The two events $|x|$ and $|y|$ are said to be independent if

$$P(|xy|) = P(|x|)\, P(|y|) \quad . \qquad (2.15)$$

A simple constructio' of the above field of probabilities is had if one assigns a probability to each elementary event or n-tuple $(a_1, \ldots a_n)$. A generalized field of probability is defined by Kolmogorov (1; p. 14) which would be of use if the above discussed probability field is extended to an infinite probability field.

Let us consider again the state equation (1.33). Let us suppose that $|Z_i|$, $|\beta_i|$ and $|{}_o\beta_i|$ are all within the same probability field for $i = 0, 1, 2, \ldots$. Then by (1.33) and (2.14) we have

$$P(Z_{i+1}) = P_{Zi}(|\beta_i|)\, P(|Z_i'|) + P_{|Zi|}(|{}_o\beta_i'|)\, P(|Z_i|) \qquad (2.16)$$

for $i = 0, 1, 2 \ldots$ where $P(|Z_o|)$ is the initial condition and where $P_{Zi}(\beta_i)$ and $P_{|Zi|}(|{}_o\beta_i'|)$ are given for $i = 0, 1, 2, 3 \ldots$. Equation (2.16) is a probability difference equation of a general two-state descrete Markov chain. If we treat (1.35) in a similar manner we obtain

$$P(|Z_{i+1}^{(j)}|) = P_{|Z_i'^{(j)}|}\left(\left|\beta^{(j)}(Z_i^{(1)}, \ldots Z_i^{(j-1)}, Z_i^{(j+1)}, \ldots Z_i^{(N)}; \omega_i^{(1)}, \ldots \omega_i^{(M)}\right|\right) P(|Z_{i1}'^{(j)}|)$$
$$+ P_{|Z_i^{(j)}|}\left(\left|{}_o\beta^{(j)}(Z_i^{(1)}, \ldots Z_i^{(j-1)}, Z_i^{(j+1)}, \ldots Z_i^{(N)}; \omega_i^{(1)}, \ldots \omega_i^{(M)}\right|\right) P(|Z_i^{(j)}|) \qquad (2.17)$$

where the initial condition is $P(|Z_o^{(j)}|)$ as the system of probability difference equations for the simple Boolean machine. The solution of (2.17) will depend on the independence of the events $|Z_i^{(j)}|$ and

(1) A. N. Kolmogorov, op cit.

$$|\beta_i^{(j)}| = |\beta^{(j)}\ Z_i^{(1)}, \ldots \left(Z_i^{(j-1)},\ Z_i^{(j+1)}, \ldots Z_i^{(N)};\ \omega_i^{(1)}, \ldots \omega_i^{(M)}\right)|$$

and $|Z_i^{(j)}|$ and

$$|{}_o\beta_i^{(j)}| = |{}_o\beta^{(j)}\ Z_i^{(1)}, \ldots \left(Z_i^{(j-1)},\ Z_i^{(j+)}, \ldots Z_i^{(N)};\ \omega_i^{(1)}, \ldots \omega_i^{(M)}\right)| \quad ;$$

and moreover, upon the question of statistical equilibrium. In (2.16) if $\lim_{i \to \infty} P(|Z_i|)$, $\lim_{i \to \infty} P_{|Z_i'|}(|\beta_i|)$ and $\lim_{i \to \infty} P_{|Z_i|}(|{}_o\beta_i'|)$ exist then we say that equation (2.16) approaches a statistical equilibrium for $i \to \infty$. If (2.16) approaches a statistical equilibrium then we will have for $P(|Z\ \ |)$

$$P(Z_\infty|) = \frac{P_{|Z_\infty'|}(|\beta_\infty|)}{P_{|Z_\infty'|}(|\beta_\infty|) + P_{|Z_\infty|}(|{}_o\beta_\infty|)} \tag{2.18}$$

Some of the mathematical answers to the questions of independence and statistical equilibrium of the system (2.17) will be of interest later. The questions are centered around $\beta_i^{(j)}$ and ${}_o\beta_i^{(j)}$, whether they are determined, i. e., $\beta_i^{(j)} = I$, or may be chosen randomly; the problem is connected with the constraints of the variables in the system, i.e. constraints are necessarily imposed in the design of counters which cycle on numbers other than 2^n.

Let us consider the following elementary example of the use of (2.16). Suppose x_i and y_i are a sequence of binary digits, either 0 or I. Let C_i be the binary carry of adding numerically x_{i-1}, y_{i-1} and C_{i-1}. Let S_i be the sum moduli 2 of x_i, y_i and C_i. Then we have

$$S_i = x_i \oplus y_i \oplus C_i = x_i'\ y_i'\ C_i + x_i'\ y_i\ C_i + x_i\ y_i'\ C_i' + x_i\ y_i\ C_i$$

$$C_{i+1} = x_i\ y_i\ C_i' + (x_i'\ y_i')'\ C_i \tag{2.19}$$

for i = 0, 1, 2. . . where C_o is the initial condition and where x_i and y_i are randomly chosen for i = 0, 1, 2. . . . Equations (2.19) are the state equations for a serial binary adder. Suppose further that the events $|x_i|$, $|y_i|$ and $|C_i|$ are mutually independent and that $P(|x_i|) = P(|x|)$ and $P(|y_i|) = P(|y_\infty|)$ all i. Then by (2.14) and (2.16) it is not difficult to show that

$$P(|C_\infty|) = \frac{P(|x_\infty|)\,P(|y_\infty|)}{P(|x_\infty|)\,P(|y_\infty|) + P(|x'_\infty|)\,P(|y'_\infty|)}$$

and

$$P(|\delta_\infty|) = P(|x_\infty|)\,P(|y_\infty|) + P(|x'_\infty|)\,P(|y'_\infty|)\left(\frac{P(|x'_\infty|)\,P(|y'_\infty|) + P(|x_\infty|)\,P(|y'_\infty|)}{P(|x_\infty|)\,P(|y_\infty|) + P(|x'_\infty|)\,P(|y'_\infty|)}\right) .$$

SOME MATHEMATICAL REMARKS ON THE BOOLEAN MACHINE

Part II

3. A Canonical Representation of the Simple Boolean System

In Section 1 of Part I the system of equations for the simple Boolean system we shall begin with are given by (1134) or

$$Z_{i+1}^{(j)} = f^{(j)}(Z_i^{(1)}, Z_i^{(2)}, \ldots Z_i^{(N)}; \omega_i^{(1)}, \omega_i^{(2)}, \ldots \omega_i^{(Ms)}) \tag{3.1}$$

where $(j = 1, \ldots N)$, $(i = 0, 1, 2, \ldots)$, initially $Z_o^{(j)}$ = (either 0 or I) for $(j = 1, \ldots N)$ and $\omega_i^{(k)}$ for $(k = 1, \ldots M)$ constitute a set of independent functions for $i = 0, 1, 2, \ldots$ By an evident change in notation which shows the dependence on the parameter t, rewrite (3.1) as

$$S_j(t_{i+1}) = f_j\left(S_1(t_i), S_2(t_i), \ldots S_N(t_i); W_1(t_i), \ldots W_M(t_i)\right) \tag{3.2}$$

where $(j = 1, \ldots N)$, $(i = 0, 1, 2, \ldots)$, initially $S_j(t_o)$ = (either 0 or I) for $(j = 1, \ldots N)$ and $W_k(t_i)$ for $(k = 1, \ldots M)$ constitute a set of independent functions for $i = 0, 1, 2, \ldots$

By de Morgan's theorem (see Section 2 or [2; p. 13]) we may expand (3.2) as

$$\begin{aligned} S_j(t_{i+1}) = {} & f_j\left(0, \ldots 0, 0; W_1(t_i), \ldots W_M(t_i)\right) S_1'(t_i) \ldots S_{N-1}'(t_i) S_N'(t_i) \\ & + f_j\left(0, \ldots 0, I; W_1(t_i), \ldots W_M(t_i)\right) S_1'(t_i) \ldots S_{N-1}'(t_i) S_N(t_i) \\ & + f_j\left(0, \ldots I, 0; W_1(t_i), \ldots W_M(t_i)\right) S_1'(t_i) \ldots S_{N-1}(t_i) S_N'(t_i) + \ldots + \\ & + f_j\left(I, \ldots I, I; W_1(t_i), \ldots W_M(t_i)\right) S_1(t_i) \ldots S_{N-1}(t_i) S_N(t_i) \; . \end{aligned} \tag{3.3}$$

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[2] Rosenbloom, P. C. The Elements of Mathematical Logic (Dover 1950).

Now (3.3) may be rewritten as

$$S_j(t_{i+1}) = \sum_{\lambda=0}^{2^N-1} a_j\left(\lambda; W_1(t_i), \ldots W_M(t_i)\right) E^{\lambda}(t_i) \tag{3.4}$$

where

$$\begin{aligned}
a_j\left(0; W_1(t_i), \ldots W_M(t_i)\right) &= f_j\left(0, \ldots 0, 0; W_1(t_i), \ldots W_M(t_i)\right) \\
a_j\left(1; W_1(t_i), \ldots W_M(t_i)\right) &= f_j\left(0, \ldots 0, I; W_1(t_i), \ldots W_M(t_i)\right) \\
a_j\left(2; W_1(t_i), \ldots W_M(t_i)\right) &= f_j\left(0, \ldots I, 0; W_1(t_i), \ldots W_M(t_i)\right) \\
&\vdots \\
a_j\left(2^N-1; W_1(t_i), \ldots W_M(t_i)\right) &= f_j\left(I, \ldots I, I; W_1(t_i), \ldots W_M(t_i)\right)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
E^0(t_i) &= S_1'(t_i) \ldots S_{N-1}'(t_i)\, S_N'(t_i) \\
E^1(t_i) &= S_1'(t_i) \ldots S_{N-1}'(t_i)\, S_N(t_i) \\
E^2(t_i) &= S_1'(t_i) \ldots S_{N-1}(t_i)\, S_N'(t_i) \\
&\vdots \\
E^{2^N-1}(t_i) &= S_1(t_i) \ldots S_{N-1}(t_i)\, S_N(t_i) \quad .
\end{aligned} \tag{3.6}$$

By (3.6) the $E^{\lambda}(t_i)$'s satisfy the following property:

$$\sum_{\lambda=0}^{2^N-1} E^{\lambda}(t_i) = I \tag{3.7}$$

and

$$\begin{aligned}
E^{\lambda}(t_i)\, E^{\mu}(t_i) &= 0 && \text{if } \lambda \neq \mu \text{ and} \\
&= E^{\mu}(t_i) && \text{if } \lambda = \mu \quad .
\end{aligned}$$

By (3.4), (3.6) and (3.7) we have

$$E^0(t_{i+1}) = S_1'(t_{i+1}) \ldots S_{N-1}'(t_{i+1}) S_N'(t_{i+1})$$

$$= \left[\sum_{\lambda=0}^{2^N-1} a_1' \;\lambda; W_1(t_i), \ldots W_M(t_i) \; E^\lambda(t_i)\right] \ldots$$

$$= \left[\sum_{\lambda=0}^{2^N-1} a_1' \;\lambda; W_1(t_i), \ldots W_M(t_i) \; E^\lambda(t_i)\right] \left[\sum_{\lambda=0}^{2^N-1} a_N' \;\lambda; W_1(t_i), \ldots W_M(t_i) \; E^\lambda(t_i)\right]$$

$$\vdots \qquad \vdots$$

$$E^{2^N-1}(t_{i+1}) = S_1(t_{i+1}) \ldots S_{N-1}(t_{i+1}) S_N(t_{i+1})$$

$$= \left[\sum_{\lambda=0}^{2^N-1} a_1 \;\lambda; W_1(t_i), \ldots W_M(t_i) \; E^\lambda(t_i)\right] \ldots$$

$$\left[\sum_{\lambda=0}^{2^N-1} a_{N-1} \;\lambda; W_1(t_i), \ldots W_M(t_i) \; E^\lambda(t_i)\right] \left[\sum_{\lambda=0}^{2^N-1} a_N \;\lambda; W_1(t_i), \ldots W_M(t_i) \; E^\lambda(t_i)\right]$$

or

$$E^\mu(t_{i+1}) = \sum_{\lambda=0}^{2^N-1} \gamma \;\mu, \lambda; W_1(t_i) \ldots W_M(t_i) \; E^\lambda(t_i) \tag{3.8}$$

where $(\mu = 0, 1, 2, \ldots 2^N-1)$,

$$\gamma\left(0,\lambda;W_1(t_i),\ldots W_M(t_i)\right) = a'_1\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)\ldots a'_{N-1}\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)$$
$$a'_N\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)$$

$$\gamma\left(1,\lambda;W_1(t_i),\ldots W_M(t_i)\right) = a'_1\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)\ldots a'_{N-1}\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)$$
$$a_N\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)$$

$$\vdots \qquad\qquad \vdots$$

$$\gamma\left(2^N-1,\lambda;W_1(t_i),\ldots W_M(t_i)\right) = a_1\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)\ldots a_{N-1}\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right)$$
$$a_N\left(\lambda;W_1(t_i),\ldots W_M(t_i)\right) \qquad (3.9)$$

and

$$\sum_{\mu=0}^{2^N-1}\left(\gamma\ \mu,\lambda;W_1(t_i),\ldots W_M(t_i)\right) = I$$

and

$$\gamma\left(\sigma,\lambda;W_1(t_i),\ldots W_M(t_i)\right)\ \gamma\left(\mu;\lambda;W_1(t_i),\ldots W_M(t_i)\right) = 0 \text{ if } \sigma \neq \mu \text{ and}$$
$$= \gamma\left(\mu,\lambda;W_1(t_i),\ldots W_M(t_i)\right) \text{ if } \sigma = \mu \qquad (3.10)$$

for $(\lambda = 0, 1, 2, \ldots 2^N-1)$.

The canonical form of the simple Boolean system, given by (3.8), may be further expanded as

$$E^\mu(t_{i+1}) = \sum_{\lambda=0}^{2^N-1}\gamma(\mu,\lambda;0,\ldots 0,0)\,E^\lambda(t_i)\,W'_1(t_i)\ldots W'_{M-1}(t_i)\,W'_M(t_i)$$
$$+\sum_{\lambda=0}^{2^N-1}\gamma(\mu,\lambda;0,\ldots 0,I)\,E^\lambda(t_i)\,W'_1(t_i)\ldots W'_{M-1}(t_i)\,W_M(t_i) + \ldots +$$
$$+\sum_{\lambda=0}^{2^N-1}\gamma(\mu,\lambda;I,\ldots I,I)\,E^\lambda(t_i)\,W_1(t_i)\ldots W_{M-1}(t_i)\,W_M(t_i)$$

or

$$E^{\mu}(t_{i+1}) = \sum_{\lambda=0}^{2^N-1} \sum_{r=0}^{2^M-1} \beta(\mu,\lambda;r)\, E^{\lambda}(t_i)\, V^{r}(t_i) \tag{3.11}$$

where

$$\begin{aligned}
\beta(\mu,\lambda;0) &= \gamma(\mu,\lambda;0,\ldots 0,0)\\
\beta(\mu,\lambda;1) &= \gamma(\mu,\lambda;0,\ldots 0,I)\\
&\vdots\\
\beta(\mu,\lambda;2^M-1) &= \gamma(\mu,\lambda;I,\ldots I,I)
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
V^0(t_i) &= W_1'(t_i)\ldots W_{M-1}'(t_i)\, W_M'(t_i)\\
V^1(t_i) &= W_1'(t_i)\ldots W_{M-1}'(t_i)\, W_M(t_i)\\
V^2(t_i) &= W_1'(t_i)\ldots W_{M-1}(t_i)\, W_M'(t_i)\\
&\vdots\\
V^{2^M-1}(t_i) &= W_1(t_i)\ldots W_{M-1}(t_i)\, W_M(t_i) \quad .
\end{aligned} \tag{3.13}$$

The canonical form of the simple Boolean system, given by (3.11) reduces to

$$E^{\mu}(t_{i+1}) = \sum_{\lambda=0}^{2^N-1} \beta(\mu,\lambda)\, E^{\lambda}(t_i) \qquad (\mu = 0, 1, 2, \ldots 2^N-1) \tag{3.14}$$

where $\beta(\mu,\lambda)$ are constants (either 0 or I), the canonical form of the simple Boolean system with no external inputs, when the functions f_j, given by (3.2), are independent of the functions

$$W_1(t_i),\ W_2(t_i),\ldots W_M(t_i) \quad .$$

Let us consider (3.11) in a form which uses the summation convention of tensor analysis. Let

$$\beta(\mu,\lambda;r) = B^{\mu}_{\lambda;r}$$

then (3.11) becomes

$$E^{\mu}(t_{i+1}) = B^{\mu}_{\lambda;r}\, E^{\lambda}(t_i)\, V^{r}(t_i) \qquad (\mu = 0,1,\ldots 2^{N}-1) \tag{3.15}$$

where

$$\begin{aligned} E^{\lambda}(t_i)\, E^{\mu}(t_i) &= E^{\mu}(t_i) && \text{when } \lambda = \mu \\ &= 0 && \text{when } \lambda \neq \mu, \\ V^{s}(t_i)\, V^{r}(t_i) &= V^{r}(t_i) && \text{when } s = r \\ &= 0 && \text{when } s \neq r \end{aligned}$$

and summations are on λ and γ for $(\lambda = 0,1,\ldots 2^{N}-1)$ and $(\gamma = 0,1,\ldots 2^{\mu}-1)$.

We are now in a position to consider the solution of the simple Boolean system (3.2) or its canonical equivalent (3.15). By the above and (1.6) of Part I we may replace in (3.5) the non-exclusive "or" operation + by $\oplus$, the Boolean ring sum operation or the exclusive "or" operation. With this observation in mind we may interpret (3.15) as a matrix difference equation

$$\mathcal{E}(t_{i+1}) = B_{v}(t_i)\, \mathcal{E}(t_i) \tag{3.16}$$

where

$$\mathcal{E}(t_i) = \begin{pmatrix} E^{0}(t_i) \\ E^{1}(t_i) \\ \cdot \\ \cdot \\ \cdot \\ E^{2^{N}-1}(t_i) \end{pmatrix} \tag{3.17}$$

$$B_v(t_i) = \left(B^{\mu}_{\lambda;r} Y^r(t_i)\right) = \begin{pmatrix} B^o_{o;r} V^r(t_i), B^o_{1;r} V^r(t_i), \ldots B^o_{2^N-1;r} V^r(t_i) \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ B^{2^N-1}_{o;r} V^r(t_i), B^{2^N-1} V^r(t_i), \ldots B^{2^N-1}_{2^N-1;r} V^r(t_i) \end{pmatrix}, \tag{3.17}$$

the initial condition of (3.16) is

$$\mathcal{E}(t_o) = \begin{pmatrix} E^0(t_o) \\ E^1(t_o) \\ \cdot \\ \cdot \\ \cdot \\ E^{2^N-1}(t_o) \end{pmatrix} \tag{3.18}$$

and the elements of the matrices $\mathcal{E}(t_o)$ and $B_v(t_i)$ for all t_i are either 0 or I, the two elements which form afield F_2 with respect to $(\cdot)$ and $+$, isomorphic to the field of residue classes, modulo 2.

The solution of (3.16) with the initial condition (3.18) is by induction,

$$\mathcal{E}(t_i) = \prod_{i=0}^{i} B_v(t_i)\ \mathcal{E}(t_o) \tag{3.19}$$

where

$$\prod_{j=0}^{i} B_v(t_j) = B_v(t_i)\, B_v(t_{i-1}) \ldots B_v(t_1)\, B_v(t_o)$$

and the products are ordinary matrix products of matrices whose elements are from F_2. The uniqueness of the solution (3.19) is evident if one considers two solutions which satisfy (3.16) and have the same initial condition. These two solutions may be shown to be identical for all t_i by induction. We, therefore, have the following theorem:

Theorem 3.1 The solution of the simple Boolean system (3.2) or its canonical equivalent (3.15) with initial condition (3.18) exists, is unique and is given explicitly by (3.19) in the notation of (3.17).

The facts leading to and the statement of Theorem 3.1 give independent mathematical support to the conclusion reached in Part I. section 1, that the simple Boolean system (3.2) may be analysed physically by a machine consisting of N-clocked flip flops for the dependent variables and suitable physical devices for producing the sums and products of the variables

$$S_1(t_i),\ S_2(t_i), \ldots S_N(t_i),\ W_1(t_i), \ldots W_M(t_i)$$

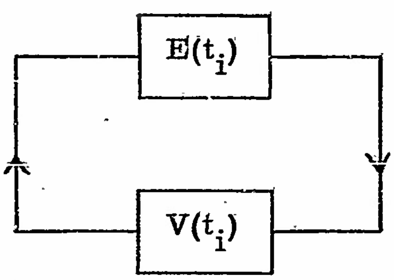
for $i = 0, 1, 2, \ldots$. We will next investigate the result of interconnecting two or more simple Boolean systems.

Let us call the simple Boolean system, given by (3.5), $E(t_i)$ and let the system, given by

$$V^r(t_{i+1}) = C^r_{\sigma;s}\, V^s(t_i)\, E^\sigma(t_i) \tag{3.20}$$

where $(r, s = 0, 1, \ldots 2^M - 1)$ and $(\sigma = 0, 1, \ldots 2^N - 1)$, be called $V(t_i)$. The system $E(t_i)$ and $V(t_i)$ are interconnected or feed into one another. This is shown by Diagram 1.

Diagram 1



If we combine (3.15) and (3.20), utilizing (3.16), we have

$$E^{\mu}(t_{i+1})\, V^{r}(t_{i+1}) = B^{\mu}_{\lambda;q}\, E^{\lambda}(t_i)\, V^{q}(t_i)\, C^{r}_{\sigma;s}\, V^{s}(t_i)\, E^{\sigma}(t_i)$$

$$= B^{\mu}_{\lambda;q}\, C^{r}_{\sigma;s} (E\ (t_i)\, E^{\lambda}(t_i)\ \ V^{q}(t_i)\, V^{s}(t_i)$$

$$= B^{\mu}_{\lambda;s}\, C^{r}_{\lambda;s}\, E^{\lambda}(t_i)\, \gamma^{s}(t_i)$$

where $\mu, \lambda = 0, 1, \ldots 2^{N}-1$ and $r, s = 0, 1, \ldots 2^{M}-1$. Suppose we let

$$G^{0}(t_i) = E^{0}(t_i)\, V^{0}(t_i) \qquad \text{and} \qquad A^{0}_{0} = B^{0}_{0;0}\, C^{0}_{0;0}$$

$$G^{1}(t_i) = E^{0}(t_i)\, V^{1}(t_i) \qquad A^{1}_{0} = B^{0}_{0,0}\, C^{0}_{0,0}$$

$$\vdots \qquad \vdots$$

$$G^{2^{M}-1}(t_i) = E^{0}(t_i)\, V^{2^{M}-1}(t_i) \qquad A^{2^{M+N}-1}_{0} = B^{2^{N}-1}_{0,0}\, C^{2^{M}-1}_{0,0}$$

$$G^{2^{M}}(t_i) = E^{1}(t_i)\, V^{0}(t_i) \qquad A^{0}_{1} = B^{0}_{0;1}\, C^{0}_{0;1}$$

$$G^{2^{M}+1}(t_i) = E^{1}(t_i)\, V^{1}(t_i) \qquad A^{1}_{1} = B^{0}_{0;1}\, C^{1}_{0;1}$$

$$\vdots \qquad \vdots$$

$$G^{2^{M+1}-1}(t_i) = E^{1}(t_i)\, V^{2^{M}-1}(t_i) \qquad A^{2^{M+N}-1}_{1} = B^{2^{N}-1}_{0;1}\, C^{2^{M}-1}_{0;1}$$

$$\vdots \qquad \vdots$$

$$G^{2^{M+N-1}-1}(t_i) = E^{2^N-1}(t_i)\, V^0(t_i) \qquad A^0_{2^{M+N}-1} = B^0_{2^N-1;2^M-1}\, C^0_{2^N-1;2^M-1}$$

$$G^{2^{M+N-1}}(t_i) = E^{2^N-1}(t_i)\, V^1(t_i) \qquad A^1_{2^{M+N}-1} = B^0_{2^N-1;2^M-1}\, C^0_{2^N-1;2^M-1}$$

$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$$

$$G^{2^{M+N}-1}(t_i) = E^{2^N-1}(t_i)\, V^{2^M-1}(t_i) \qquad A^{2^{M+N}-1}_{2^{M+N}-1} = B^{2^N-1}_{2^N-1;2^M-1}\, C^{2^M-1}_{2^N-1;2^M-1}$$

then (3.21) may be rewritten as

$$G^e(t_{i+1}) = A^e_m\, G^m(t_i) \tag{3.22}$$

where $m, e = 0, 1, 2, \ldots 2^{M+N}-1$. By (3.15) we see that the two interconnected simple Boolean systems $E(t_i)$ and $V(t_i)$ form a higher order simple Boolean system $G(t_i)$, given by (3.22). By assuming that n-interconnected simple Boolean systems form a simple Boolean system $H_n(t_i)$ of higher order for purposes of induction, we see by the above argument that $H_n(t_i)$ interconnected with another simple Boolean system $A(t_i)$ will again form a simple Boolean system $H_{n+1}(t_i)$ of higher order. Thus our induction hypothesis is complete and we have the following theorem.

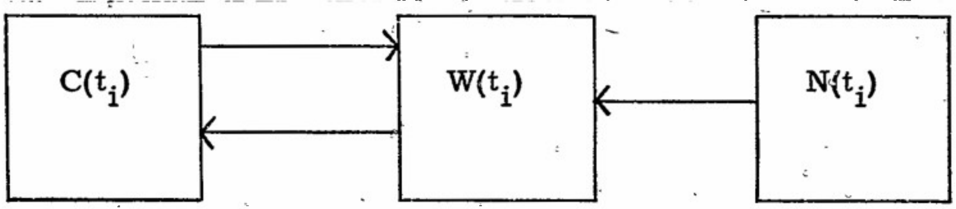
Theorem 3.2 The system of n-interconnected simple Boolean systems of t_i is a simple Boolean system of t_i for $n = 2, 3, 4 \ldots$

By (1.31) of Section 1, Part I, the equations of the clocked flip flop form a simple Boolean system (of two configurations). Therefore, by Theorem 3.2 any number of interconnected two configuration Boolean systems of t_i will compose a simple Boolean system of t_i. Combining this statement with the statement, following (1.35) of Section 1 of Part I we have the following theorem.

Theorem 3.3 The necessary and sufficient condition that a system of Boolean difference equations of one discrete parameter t_i form a simple Boolean system is that this system may be physically represented by a machine which consists only of clocked flip flops and suitable physical devices for producing the sums and products of the outputs of the flip flops.

Another consequence of Theorem 3.2 is a conceptual approach to the so-called digital control problem. Suppose $C(t_i)$ is a simple Boolean system whose inputs are from the local external world $W'(t)$ (through digitalized instruments) and that $C(t_i)$ delivers outputs to the local external world $W'(t)$ through digital actuators (motors, hydraulic pistons, etc.). Suppose further that $W'(t)$ has well defined laws which are corrupted in general by a noise source $N'(t)$ from the world in the large. Suppose that the laws of $W'(t)$ s action and reaction with respect to the system $C(t_i)$ may be approximated to some degree of accuracy by a simple Boolean system $W(t_i)$ and that $N'(t)$ May be approximated by $N(t_i)$. Diagram 2 shows the interconnections of $C(t_i)$, $W(t_i)$ and $N(t_i)$.

Diagram 2



By Theorem 3.2 we may regard the total system comprising $C(t_i)$, $W(t_i)$, $N(t_i)$ and their interconnections as a simple Boolean system. Theoretically this system has a solution as was demonstrated by (3.19). In the next section a further understanding of this system will be obtained by the use of probability theory.

4. The Markovian nature of the simple Boolean system

By the rules of probability, given by I, II, III, (2.14) and (2.15) of Section 2, Part I, we have for the simple Boolean system by (3.15) and (3.16),

$$\begin{aligned} P\left(\left|E^{\mu}(t_{i+1})\right|\right) &= b^{\mu}_{\lambda;r} P\left(\left|E^{\lambda}(t_i)\, V^{r}(t_i)\right|\right) \\ &= b^{\mu}_{\lambda;r} P\left(\left|E^{\lambda}(t_i)\right|\right) P_{\left|E^{\lambda}(t_i)\right|}\left(\left|V^{r}(t_i)\right|\right) \end{aligned} \tag{4.1}$$

for $(\mu, \lambda = 0, 1, \ldots 2^N - 1)$ and $(r = 0, 1, 2, \ldots 2^M - 1)$ where $b^{\mu}_{\lambda;r}$ = the integer 1 when $B^{\mu}_{\lambda;r} = I$ and 0 when $B^{\mu}_{\lambda;r} = 0$ (this assumes that $B^{\mu}_{\lambda;r}$ for each triplet (λ, μ, r) is either the constant I or 0 for all t_i). If we assume the conditional probabilities

$$P_{|E^{\lambda}(t_i)|}\left(|V^r(t_i)|\right)$$

are known and construct the matrices

$$e(t_i) = \begin{pmatrix} P\left(|E^0(t_i)|\right) \\ P\left(|E^1(t_i)|\right) \\ \cdot \\ \cdot \\ \cdot \\ P\left(|E^{2^N-1}(t_i)|\right) \end{pmatrix} \qquad \text{and}$$

$$T_v(t_i) = \begin{pmatrix} b^0_{0;r} P_{|E^0(t_i)|}\left(|V^r(t_i)|\right) & \ldots & b^0_{2^N-1;r} P_{|E^{2^N-1}(t_i)|}\left(|V^r(t_i)|\right) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b^{2^N-1}_{0;r} P_{|E^0(t_j)|}\left(|V^r(t_i)|\right) & \ldots & b^{2^N-1}_{2^N-1;r} P_{|E^{2^N-1}(t_i)|}\left(|V^r(t_i)|\right) \end{pmatrix} \tag{4.2}$$

then (4.1) becomes the matrix equation

$$e(t_{i+1}) = T_v(t_i)\, e(t_i) \quad , \tag{4.3}$$

$(i = 0, 1, 2, \ldots)$ with the assumed initial condition,

$$e(t_o) = \begin{pmatrix} P\left(\left|E^0(t_o)\right|\right) \\ P\left(\left|E^1(t_o)\right|\right) \\ \cdot \\ \cdot \\ \cdot \\ P\left(\left|E^{2^N-1}(t_o)\right|\right) \end{pmatrix}$$

The unique solution of (4.3) is easily shown to be

$$e(t_i) = \prod_{j=0}^{i} T_V(t_i)\, e(t_o) \tag{4.4}$$

where

$$\prod_{j=0}^{i} T_V(t_i) = T_V(t_i)\, T_V(t_{i-1}) \ldots T_V(t_i)\, T_V(t_o) \quad .$$

If the conditional probabilities $P_{\left|E^\lambda(t_i)\right|}\left(\left|V^r(t_i)\right|\right)$ are the same for all t_i, then

$$T_V = T_V(t_o) = T_V(t_i) = \ldots \quad . \tag{4.5}$$

With (4.5) the solution (4.4) may be expressed as

$$e(t_i) = T_V^i\, e(t_o) \tag{4.6}$$

where T_V^i is the i-th power of the matrix T_V. If the events $\left|V^r(t_i)\right|$ are independent of each event $\left|E^\lambda(t_i)\right|$, then

$$P_{\left|E^\lambda(t_i)\right|}\left(\left|V^r(t_i)\right|\right) = P\left(\left|V^r(t_i)\right|\right) \tag{4.7}$$

for $(i = 0,1,2\ldots)$. If (4.7) is true, then the simple Boolean system $E(t_i)$ is said to be independent of its input system $V(t_i)$. If the system $E(t_i)$ has no input system $V(t_i)$, then

$$T_v = T \tag{4.8}$$

where T is a matrix whose elements are either the integer 1 or 0. (4.8) is the case of the simple Boolean system (3.14) with no external inputs.

Let us consider the system of Diagram 2. The system of Boolean difference equations for $C(t_i)$ and $W(t_i)$ with input $N(t_i)$ will be

$$C^{\mu}(t_{i+1}) = J^{\mu}_{\lambda;q}\, C^{\lambda}(t_i)\, W^{q}(t_i) \tag{4.9}$$

and

$$W^{r}(t_{i+1}) = K^{r}_{\sigma;s;p}\, C\,(t_i)\, W^{s}(t_i)\, N^{s}(t_i)$$

for $(\lambda,\mu,\sigma = 0,1,\ldots 2^{N}-1)$, $(q,r,s = 0,1,\ldots 2^{M}-1)$ and $(p = 0,1,\ldots 2^{r}-1)$. If we combine systems $C(t_i)$ and $W(t_i)$ as in (3.21) we obtain

$$C^{\mu}(t_{i+1})\, W^{r}(t_{i+1}) = J^{\mu}_{\lambda;s}\, K^{r}_{\lambda;s;p}\, N^{s}(t_i)\, C^{\lambda}(t_i)\, W^{s}(t_i) \quad . \tag{4.10}$$

The total system (4.10) of $C(t_i)$, $W(t_i)$ and $N(t_i)$ may be put, using the argument preceding (3.22), in the form of system (3.22). Then we have

$$U^{\ell}(t_{i+1}) = D^{\ell}_{m}\,[N^{\rho}(t_i)]\, U^{m}(t_i) \tag{4.11}$$

where the initial condition is $U^{m}(t_o)$, the matrix $D^{\ell}_{m}\,[N^{\rho}(t_i)]$ denotes the functional dependence of the elements of the matrix on $N^{\rho}(t_i)$ for $(\rho = 0,1,\ldots 2^{r}-1)$ and $m,\ell = 0,1,2,\ldots 2^{M+N}-1)$. Let us suppose that the approximate noise source $N(t_i)$ is unaffected by the system $U(t_i)$. That is, the events $|N^{\rho}(t_i)|$ are independent of $|U^{m}(t_i)|$ all ρ and m. Moreover, let

$$P(|N^{\rho}|) = P(|N^{\rho}(t_o)|) = P(|N^{\rho}(t_i)|) = \ldots$$

for $(\rho = 0,1,\ldots 2^{r})$. Then as in (4.1), using (4.7), we obtain from (4.11),

$$P(|U^{\ell}(t_{i+1})|) = d^{\ell}_{m}\,[P(|N^{\rho}|)]\, P(|U^{m}(t_i)|) \quad , \tag{4.12}$$

for $(\ell, m = 0,1,\ldots 2^{M+N}-1)$ where for each ℓ and m $d_m^{\ell}[P(|N^{\rho}|)]$ is either 0 or a partial sum of the probabilities $P(|N^{\rho}|)$ for $(\rho = 0,1,\ldots 2^r-1)$ and the initial condition is chosen to be $P(|U^m(t_o)|)$. We may rewrite (4.12) in the form of (4.3) or as the matrix equation

$$U(t_{i+1}) = T_N\, U(t_i) \tag{4.13}$$

where the initial condition is $U(t_o)$,

$$U(t_i) = \begin{pmatrix} P(|U^0(t_i)|) \\ P(|U^1(t_i)|) \\ \cdot \\ \cdot \\ \cdot \\ P(|U^{2^{M+N}-1}(t_i)|) \end{pmatrix},$$

$i = 0,1,2,\ldots)$ and

$$T_N = \begin{pmatrix} d_0^0[P(|N^{\rho}|)] & \ldots & d^0_{2^{M+N}-1}[P(|N^{\rho}|)] \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ d_0^{2^{M+N}-1}[P(|N^{\rho}|)] & \ldots & d^{2^{M+N}-1}_{2^{M+N}-1}[P(|N^{\rho}|)] \end{pmatrix}.$$

By (4.5) and (4.6) equation (4.13) has the solution

$$U(t_i) = T_N^i\, U(t_o) \tag{4.14}$$

for $(i = 0,1,2,3\ldots)$. From (4.14) we have obtained $P(|U^{\ell}(t_i)|)$ for $(\ell = 0,1,\ldots 2^{M+N}-1)$ and $(i = 0,1,2\ldots)$ or $P(|C^{\mu}(t_i)\, W^r(t_i)|)$ for $(\mu = 0,1,\ldots 2^N-1)$, $(r = 0,1,\ldots 2^M-1)$ and

$(i = 0,1,2,\ldots)$. Since

$$C^{\mu}(t_i) = C^{\mu}(t_i)\, I = C^{\mu}(t_i) \sum_{r=0}^{2^M-1} W^r(t_i)$$

$$= \sum_{r=0}^{2^M-1} C^{\mu}(t_i)\, W^r(t_i)$$

where

$$[C^{\mu}(t_i)\, W^r(t_i)]\ [C^{\mu}(t_i)\, W^s(t_i)] = 0 \quad \text{if } r \neq s \quad \text{and}$$

$$= C^{\mu}(t_i)\, W^s(t_i) \quad \text{if} \quad \mu = s,$$

we have by (2.14) of Part I

$$P(|C^{\mu}(t_i)|) = \sum_{r=0}^{2^M-1} P(|C^{\mu}(t_i)\, W^r(t_i)|). \tag{4.15}$$

In a similar fashion we obtain

$$P(|W^r(t_i)|) = \sum_{\mu=0}^{2^N-1} P(|C^{\mu}(t_i)\, W^r(t_i)|) \quad . \tag{4.16}$$

From (2.14) of Part I we have

$$P(|C^{\mu}(t_i)\, W^r(t_i)|) = P_{|C^{\mu}(t_i)|}(|W^r(t_i)|)\, P(|C^{\mu}(t_i)|)$$

$$= P_{|W^r(t_i)|}(|C^{\mu}(t_i)|)\, P(|W^r(t_i)|) \quad . \tag{4.17}$$

From (4.15), (4.16) and (4.19) we obtain

$$P_{|C^{\mu}(t_i)|}(|W^r(t_i)|) = \frac{P(|C^{\mu}(t_i)\, W(t_i)|)}{\sum_{s=0}^{2^M-1} P(|C^{\mu}(t_i)\, W^s(t_i)|)}$$

and

$$P_{|W^r(t_i)|}(|C^{\mu}(t_i)|) = \frac{P(|C^{\mu}(t_i)\, W^r(t_i)|)}{\sum\limits_{\sigma=0}^{2^N-1} P(|C^{\sigma}(t_i)\, W^r(t_i)|)} \tag{4.18}$$

as the conditional probabilities, relating the configurations of $C(t_i)$ and $W(t_i)$ of Diagram 2 at t_i, in terms of the probabilities of the total system

$$P(|E^{\mu}(t_i)\, V^r(t_i)|),$$

obtained as the solution of (4.13).

It is possible to utilize the methods, leading to (4.18), to obtain probabilities

$$P_{|W^r(t_i)|}(|C^{\mu}(t_{i+j})|),\quad P_{|W^r(t_i)|}(|W^s(t_{i+j})|),\quad \text{etc.}$$

not enter this phase of the problem at the present time. We have shown by (4.13) that the very general simple Boolean system, given by Diagram 2, is a discrete Markov chain or process where the transition probabilities are constants for all t_i. Since the literature is prevalent with studies of such chains, it is expected that the probability approach to the Boolean system will lead to new and possibly practical results in the subject of digital computers. For the remainder of this section we will discuss briefly the solution of Markov chains by matrices of generating functions.

The matrix solutions (4.6) and (4.14) arise from a matrix equation of the form

$$U(t_{i+1}) = P\,U(t_i) \tag{4.19}$$

where the initial condition is $U(t_o)$,

$$P = \begin{pmatrix} P_{11}, P_{12}, \cdots P_{M} \\ P_{21}, P_{22}, \\ \cdot \\ \cdot \\ \cdot \\ P_{M1}, \cdots P_{MN1} \end{pmatrix}$$

for $(j,k = 1,\ldots M)$,

$$U(t_i) = \begin{pmatrix} U_1(t_i) \\ U_2(t_i) \\ \cdot \\ \cdot \\ \cdot \\ U_M(t_i) \end{pmatrix}$$

and

$$\sum_{j=1}^{M} P_{jk} = 1 \text{ and } P_{jk} \geqslant 0 \tag{4.20}$$

for $(j,k = 1,\ldots M)$.

The unique solution of (4.19) is

$$U(t_i) = P^i U(t_o) \tag{4.21}$$

for $(i = 0,1,2\ldots)$.

Our purpose will be to discuss briefly another representation of P^i and its value for large values of i.

Now $U(t_i)$ may be considered as a vector in the M-dimensional complex column vector space C_M, composed of elements

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_M \end{pmatrix} \tag{4.22}$$

where x_i are complex numbers. Let us introduce a norm of x in (4.22) as

$$\|x\| = \sum_{i=1}^{M} |x_i| \tag{4.23}$$

With $\|x\|$ defined by (4.23), it is not difficult to show

$$\|x + y\| \leq \|x\| + \|y\|$$

and

$$\|\alpha x\| = |\alpha| \; \|x\| \tag{4.24}$$

where $x, y \,\mathcal{E}\, C_M$ and α is a complex number. Let us define the norm of an $M \times M$ matrix A, whose elements are complex numbers as

$$\|A\| = \sup_{\|\bar{x}\|=1} \|A\bar{x}\| \tag{4.25}$$

where $\bar{x} \,\mathcal{E}\, C_M$ and

$$\sup_{\|x\|=1} \|Ax\|$$

is the least upper bound or supremum of $\|Ax\|$ for x on the unit "sphere" or $\|x\| = 1$. From (4.24) and (4.25) we have for the $M \times M$ matrices A and B,

$$\|Ax\| \leq \|A\| \; \|x\| \quad ,$$

$$\|A + B\| \leq \|A\| + \|B\| \quad ,$$

$$\|AB\| \leq \|A\| \; \|B\| \qquad \text{and}$$

$$\|\alpha B\| = |\alpha| \; \|B\| \tag{4.26}$$

where α is a complex number. Let us call an $M \times M$ matrix with property (4.20) a Markov matrix.

Let $P = (p_{ij})$ be an $(M \times M)$-Markov matrix and $x \,\mathcal{E}\, C_M$ such that $\|x\| = 1$. Then

$$\|Px\| = \sum_{i=1}^{M} \Big|\sum_{j=1}^{M} p_{ij} x_j\Big| \quad \sum_{i=1}^{M} \sum_{j=1}^{M} p_{ij} |x_j| = \sum_{j=1}^{M} |x_j| \sum_{i=1}^{M} p_{ij} = \sum_{j=1}^{M} |x_j| = \|x\| = 1 \; . \tag{4.27}$$

Moreover, if x is such that

$$\sum_{i=1}^{M} x_i = 1 \qquad \text{with } x_i \geq 0$$

$$\|Px\| = \sum_{i=1}^{M} \Big|\sum_{j=1}^{M} p_{ij} x_j\Big| = \sum_{i=1}^{M} \sum_{j=1}^{M} p_{ij} x_j = \sum_{j=1}^{M} x_j \sum_{i=1}^{M} p_{ij} = \sum_{j=1}^{M} x_j = \|x\| = 1 \tag{4.28}$$

By (4.27), (4.28) and (4.25) we thus have

$$\|P\| = 1 \tag{4.29}$$

when P is a Markov matrix. Suppose $Q = (q_{ij})$ is another $(M \times M)$-Markov matrix, then the elements of PQ are

$$\sum_{k=1}^{M} p_{ik} q_{kj} \geq 0$$

for $(i,j = 1,2,\ldots M)$, and

$$\sum_{i=1}^{M} \left(\sum_{k=1}^{M} p_{ik} q_{kj} \right) = \sum_{k=1}^{M} q_{kj} = 1$$

for $(j = 1,\ldots,M)$.

Thus (4.20) is satisfied and PQ is a Markov matrix. By (4.29) we therefore have

$$1 = \|PQ\| = \|P\| \; \|Q\| \quad . \tag{4.30}$$

The generalization of (4.30) to any number of products of Markov matrices is clearly evident.

Let us now consider the series

$$I + \sum_{i=1}^{\infty} P^i s^i \quad , \tag{4.31}$$

where P is a Markov matrix and s is a complex variable. For (4.31) we have by (4.26) and (4.30)

$$\|I + \sum_{i=1}^{\infty} P^i s^i\| \quad 1 + \sum_{i=1}^{\infty} \|P^i\| \; |s|^i = \sum_{i=b}^{\infty} |s|^i = \frac{1}{1-|s|} \text{ for } |s| < 1. \tag{4.32}$$

Moreover, the matrix series (4.31) satisfies the matrix equation,

$$X(I - sP) = I \quad .$$

By (4.32) we have that (4.31) exists for $|s| < 1$ and as a consequence by the preceding sentence we have finally

$$(I - sP)^{-1} = I + \sum_{i=1}^{\infty} P^i s^i \text{ for } |s| < 1 \quad . \tag{4.33}$$

Now from matrix theory we know

$$(I - sP)^{-1} = \frac{1}{|I - sP|} \operatorname{Adj}(I - sP) = \frac{1}{|I - sP|} \left(g_{ij}(s)\right) = \left(f_{ij}(s)\right) \tag{4.34}$$

where $\operatorname{Adj}(I - sP)$ is the adjoint of the matrix $I - sP$ or the matrix of $(M - 1)$-order cofactors of $I - sP$ and $|I - sP|$ is the determinant of $I - sP$. By definition we have that $g_{ij}(s)$ and $|I - sP|$ are polynomials of s such that the degree of $g_{ij}(s)$ is less than the degree of $|I - sP|$ or

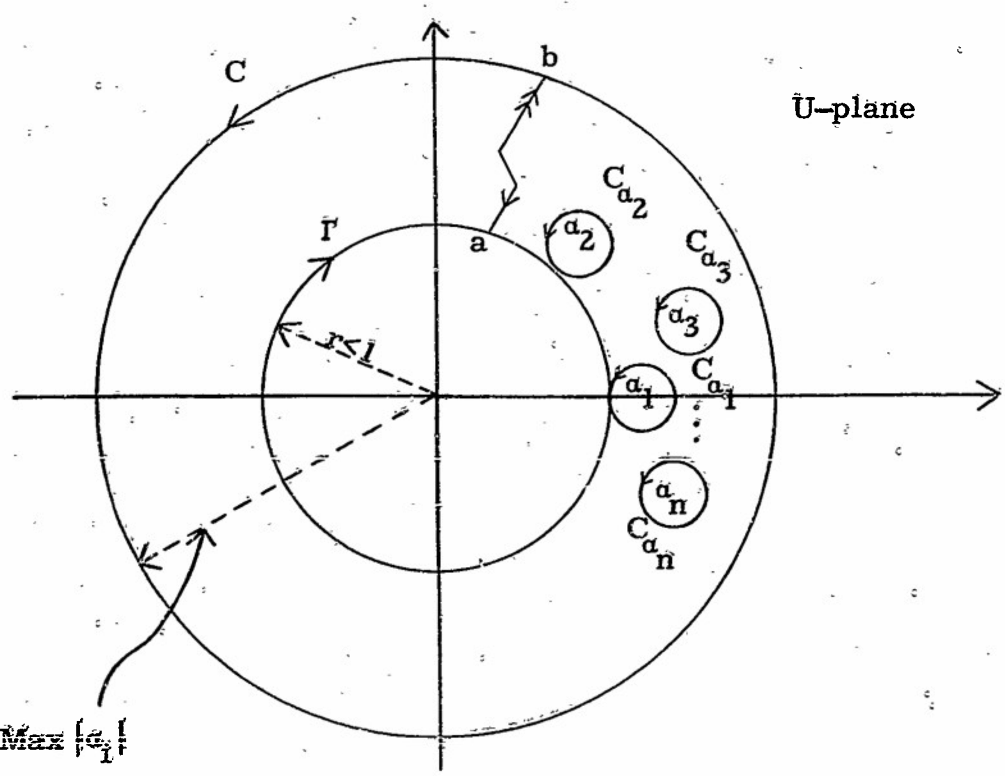
$$\deg[g_{ij}(s)] \leq \deg[|I - sP|] - 1 \quad . \tag{4.35}$$

Let us suppose that the roots of $|I - sP|$ are $a_1, a_2, \ldots a_N$ of orders $m_1, m_2, \ldots m_N$, respectively. From (4.34) this is equivalent to the statement that $f_{ij}(s)$ for $(i, j = 1, 2, \ldots, M)$ are rational fractions with possible poles at $a_1, a_2, \ldots a_N$ of orders of at most $m_1, m_2, \ldots m_N$, respectively. From (4.33) and (4.34) we have

$$|a_i| \geq 1$$

for $(i = 1, \ldots N)$.

Diagram 2



Let us consider Diagram 2. The double path from a to b in the complex U-plane is such that it does not cut across any pole α_i. The path: a to b; around C, clockwise, to b; b to a; around Γ, counterclockwise to a: is a closed counterclockwise path, call it Δ, which contains all the poles of $f_{ij}(s)$ for $(i, j = 1, \dots M)$. Thus for $|s| < r$ we have

$$\frac{1}{2\pi i}\int_{\Delta}\frac{f_{ij}(u)}{s-u}\,du = \sum_{k=1}^{N}\frac{1}{2\pi i}\int_{C_{\alpha_k}}\frac{f_{ij}(u)}{s-u}\,du \quad , \tag{4.36}$$

$(i, j = 1, \dots M)$ where C_{α_k} are small circles about α_k such that no two of these circles overlap or intersect with Γ, C or the path from a to b. Since the line integrals of $f_{ij}(u)/s-u$ from a to b and from b to a cancel, we have

$$\frac{1}{2\pi i}\int_{\Delta}\frac{f_{ij}(u)}{s-u}\,du = \frac{1}{2\pi i}\int_{\Gamma}\frac{f_{ij}(u)}{s-u}\,du + \frac{1}{2\pi i}\int_{C}\frac{f_{ij}(u)}{s-u}\,du \tag{4.37}$$

for $|s| < r$. By (4.35) $f_{ij}(u)/s-u$ has no pole at infinity and consequently no poles outside C. By Cauchy's theorem we then have

$$\frac{1}{2\pi i}\int_{C}\frac{f_{ij}(u)}{s-u}\,du = 0$$

and

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f_{ij}(u)}{s-u}\,du = f_{ij}(s)$$

which, combined with (4.36) and (4.37), give finally

$$f_{ij}(s) = \sum_{k=1}^{N}\frac{1}{2\pi i}\int_{C_{\alpha_k}}\frac{f_{ij}(u)}{s-u}\,du \tag{4.38}$$

for $|s| < r$.

From (4.38) we obtain

$$f_{ij}(s) = -\sum_{k=1}^{N} \frac{1}{2\pi i} \int_{C_{\alpha_k}} \frac{f_{ij}(u)}{u(1 - s/u)}\, du = -\sum_{k=1}^{N} \frac{1}{2\pi i} \int_{C_{\alpha_k}} \frac{f_{ij}(u)}{u} \left[\sum_{\ell=0}^{\infty} \left(\frac{s}{u}\right)^{\ell}\right] du$$

$$= -\sum_{\ell=0}^{\infty} \left[\sum_{k=1}^{N} \frac{1}{2\pi i} \int_{C_{\alpha_k}} \frac{f_{ij}(u)}{u^{\ell+1}}\, du\right] s^{\ell} \tag{4.39}$$

since $|s| < |u|$. From (4.39) the coefficient of s^{ℓ} is

$$-\sum_{k=1}^{N} \frac{1}{2\pi i} \int_{C_{\alpha_k}} \frac{f_{ij}(u)}{u^{\ell+1}}\, du = -\sum_{k=1}^{N} \frac{1}{2\pi i} \int_{C_{\alpha_k}} \left[\frac{f_{ij}(u)\,(u - \alpha_k)^{m_k}}{u^{\ell+1}}\right] \frac{du}{(u - \alpha_k)^{m_i}}\ .$$

But $f_{ij}(u)\,(u - \alpha_k)^{m_i} u^{-\ell-1}$ is regular within the circle C_{α_k} since the order of pole α_n of $f_{ij}(u)$ is at most m_i. Thus if we use the Cauchy integral formula for the n-th derative, we obtain

$$-\sum_{k=1}^{N} \frac{1}{2\pi i} \int_{C_{\alpha_k}} \frac{f_{ij}(u)}{u^{\ell+1}}\, du = -\sum_{k=1}^{N} \frac{1}{(m_{k-1})!} \frac{d^{m_{k-1}}}{(du)^{m_{k-1}}} \left\{\left[\frac{f_{ij}(u)\,(u - \alpha_k)^{m_k}}{u^{\ell+1}}\right]\right\}_{u=\alpha_k} \tag{4.40}$$

Combining (4.33), (4.34), (4.39) and (4.40) we obtain finally

$$P^{\ell} = -\sum_{k=1}^{N} \frac{1}{(m_{k-1})} \left\{\frac{d^{m_{k-1}}}{(du)^{m_{k-1}}} \left[\frac{(u - \alpha_k)^{m_k}}{|I - uP|\, u^{\ell+1}}\, \mathrm{Adj}(I - uP)\right]\right\}_{u=\alpha_k} \tag{4.41}$$

where the derivative of a matrix of functions is the matrix of the derivatives of the functions.

Another interpretation of the roots α_i of $|I - sP|$ is that they are the values of s (the proper values) for which the vector equations

$$x = sPx \qquad \text{and} \qquad y = syP$$

have non-trivial solutions (solutions other than the zero vector) where y is an M-columned row vector. Since the row vector

$$y_I = (1,1,\ldots 1) \tag{4.42}$$

satisfies

$$y = yP \quad ,$$

it is immediately evident that $s = 1$ is always one of the roots α_i, say α_1, when P is a Markov matrix. Any other roots of $|I - sP|$ of modulus one would necessarily be roots of unity.

If $s = 1$ is a first order root of $|I - sP|$, then the linear manifold generated by the solutions of $y - yP = 0$ is one dimensional and we may take the row vector y_I of (4.42) as the basic vector. Every element of this linear manifold will be of the form αy_I where α is a complex number. From (4.34) we have

$$|I - sP|\, I = \left(\sum_{j=1}^{M} g_{\ell j}(s)\,[\delta_{jk} - sp_{jk}] \right) \quad , \qquad (\ell, k = 1,\ldots M)$$

or

$$0 = \left(\sum_{j=1}^{M} g_{\ell j}(1)\,[\delta_{jk} - sp_{jk}] \right) = g_\ell - g_\ell P \tag{4.43}$$

where $\delta_{jk} = 0$ when $j \neq k$ and $= 1$ when $j = k$ and

$$g_\ell = \left(g_{\ell_1}(1),\ g_{\ell_2}(1), \ldots g_{\ell_M}(1) \right) \quad .$$

By (4.43) we see that g_ℓ is a solution to $y - yP = 0$, hence

$$g_\ell = \alpha_k y_I \qquad (\ell = 1,\ldots M)$$

where α_k are complex numbers and as a consequence

$$g_{\ell_1}(1) = g_{\ell_2}(1) = \ldots = g_{\ell_M}(1) \quad , \tag{4.44}$$

$(\ell = 1,\ldots M)$.

If all the poles of $1/|I - sP|$ are of first order, then (4.41) becomes (compare [3])

$$P^j = -\sum_{k=1}^{N} \frac{1}{\alpha_k^{j+1}\left[\frac{d}{du}\,|I - uP|\right]_{u=\alpha_k}} \mathrm{Adj}(I - \alpha_k P) = -\sum_{k=1}^{N} \frac{1}{\alpha_k^{j+1}\left[\frac{d}{du}\,|I - uP|\right]_{u=\alpha_k}} \left(g_{ij}(\alpha_k)\right)$$

$$= \sum_{k=1}^{N} \frac{1}{\alpha_k^{j+1}} Ak \tag{4.45}$$

where $\alpha_1 = 1$. By (4.44) it is evident that the elements of any given row of A, are equal. If $|\alpha_k| > 1$ for $(k = 2, \ldots M)$, then

$$\lim_{j \to \infty} P^j = A_1 \quad . \tag{4.46}$$

If

$$\alpha_1 = 1,\ \alpha_2 = \omega,\ \alpha_3 = \omega^2, \ldots \alpha_N = \omega^N - 1$$

where ω is an N-th root of unity, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} P^j = A_1 \quad , \tag{4.47}$$

since

$$\lim_{N \to \infty} \left| \sum_{j=1}^{N} \frac{\alpha^j}{N} \right| = \lim_{N \to \infty} \left| \frac{\alpha - \alpha^{N+1}}{(1 - \alpha)N} \right| = 0$$

where α is an N-th root of unity, not equal to one. We shall use (4.45), (4.46) and (4.47) in the next section. In this section we have shown the Markovian nature of the Boolean system. We have discussed briefly some elementary notions about discrete Markov chains. ~~We hope to refine our methods with further elaboration~~ in a later report.

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[3] Feller, W., Probability Theory and its Applications (John Wiley 1950).

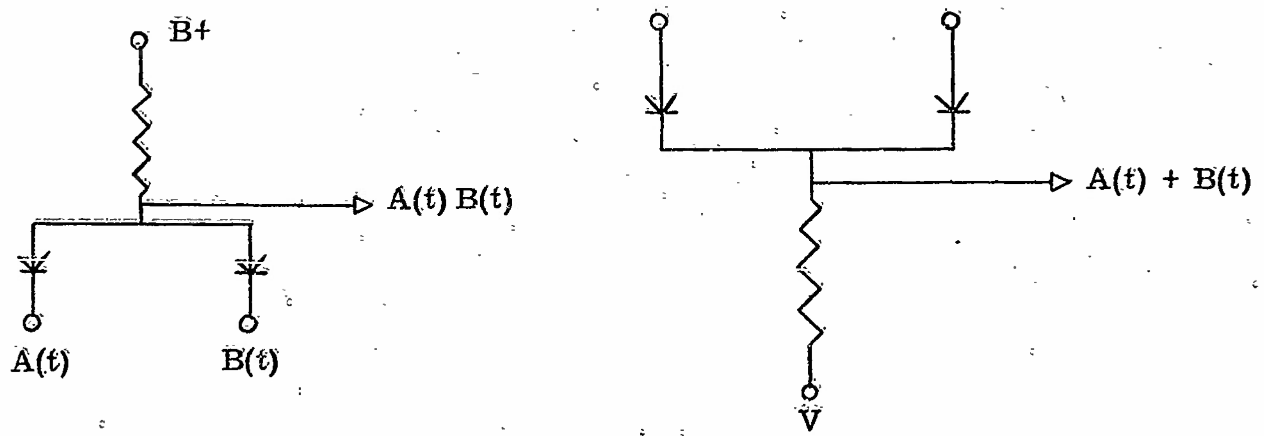
5. The Trinary Counter

Before we consider the trinary counter, let us consider briefly examples of physical devices which produce the sum and product of two Boolean functions of time and solve the flip-flop equation (1.14) of Part I. There are two schemes for producing sums and products of Boolean functions with unidirectional current devices. Scheme I is well known and discussed elsewhere [4]. Scheme II has been used for clipping voltages in various electronic circuits, but to this author's knowledge, it has not been considered seriously as a means of producing logical sums and products of two valued voltage functions in a computer.

Let us suppose that the two possible values of the Boolean time function $A(t)$ are two voltage levels, say E_H and E_L where $E_H > E_L$. Then we have $1 \equiv E_H$ and $0 \equiv E_L$. If we have $A(t)$ and $B(t)$ of the same nature then $A(t) + B(t)$ and $A(t)\,B(t)$ may be obtained by either of the two schemes, given in Diagram 3, with diodes where $B > E_H$ and $V < E_L$. The output voltage in Diagram 3 may be obtained from

Diagram 3

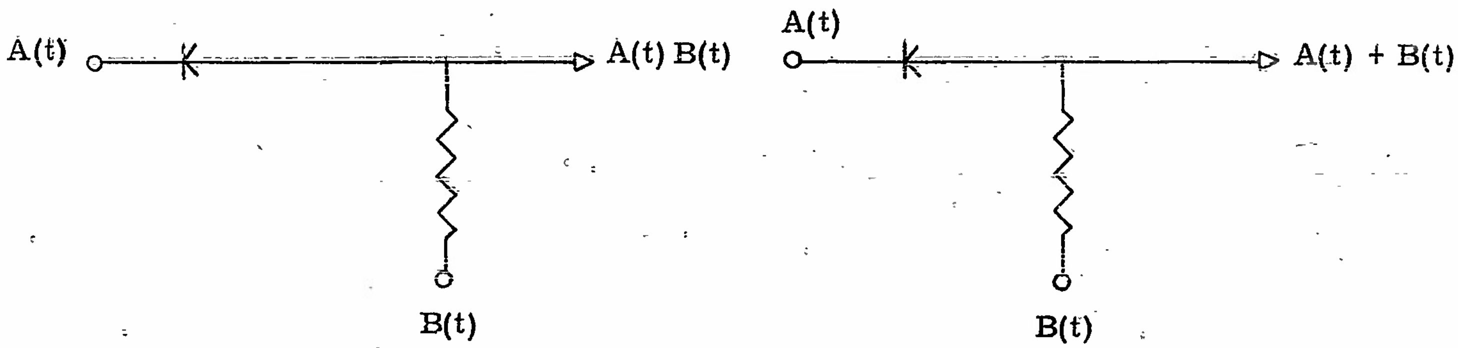
Scheme I



[4] Sprague, R. E., Techniques in The Design of Digital Computers, a paper presented at Association of Computing Machinery (March 1951).

Diagram 3 (cont.)

Scheme II



$$A(t)\ B(t) = E_L + \frac{|A(t) - E_L|\ |B(t) - E_L|}{E_H - E_L}$$

and

$$A(t) + B(t) = E_H - \frac{|E_H - A(t)|\ |E_H - B(t)|}{E_H - E_L} \tag{5.1}$$

where on the right + and − are addition and subtraction signs between real numbers. The terms on the right of (5.1) may be taken as the definitions of the "and" and "or" operations on the left. In this case one may use the rules of elementary algebra to obtain the output voltages of a diode net, composed of trees or chains of the circuits of Diagram 3.

If the voltages A(t) and B(t) are replaced by two arbitrary voltage signals f(t) and g(t) such that

$$V \leq f(t),\ g(t) \leq B+ ,$$

then the output of the "and" gates, given in Diagram 3, will be Min $\big(f(t), g(t)\big)$ and the output of the "or" gates will be Max $\big(f(t), g(t)\big)$. If we let

$$[f \wedge g](t) = \text{Min}\ \big(f(t), g(t)\big)$$

and

$$[f \vee g](t) = \text{Max}\ \big(f(t), g(t)\big)$$

for t contained in some range R, it is known that the class of continuous functions from a distributive lattice under v and $\wedge$, a partially ordered set with an upper and lower bound, satisfying the dual distributive laws,

$$(f \wedge g) \vee h = (f \vee h) \wedge (g \vee h)$$

and

$$(f \vee g) \quad h = (f \wedge h) \vee (g \wedge h) \quad .$$

This fact may be useful in the study of non-linear circuit analysis.

Let us now consider the ideal flip-flop equation (1.14) of Part I

$$X(t) = b(t)\,X'(t-) + {}_ob'(t)\,X(t) \quad , \tag{5.2}$$

where b(t) and ${}_ob(t)$ are upspike B_o-functions.

Equation (5.2) is equivalent to

$$X'(t) = b'(t)\,X(t-) + {}_ob(t)\,X'(t) \quad . \tag{5.3}$$

If we let

$$X'(t) = S(t),\ b'(t) = \beta(t),\ {}_ob'(t) = {}_o\beta(t)$$

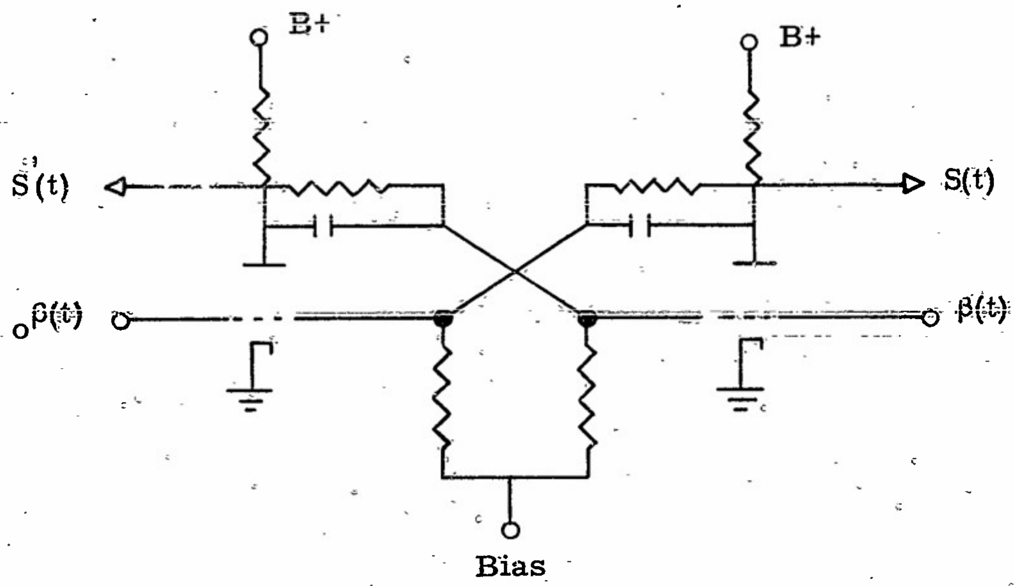
then (5.3) becomes

$$S(t) = \beta(t)\,S'(t-) + {}_o\beta(t)\,S(t-) \tag{5.4}$$

where $\beta(t)$ and ${}_o\beta(t)$ are downspike B_o-functions. Equation (5.4) is the same form as the ideal flip-flop equation (5.2), but with its two B_o-function inputs $\beta(t)$ and ${}_o\beta(t)$ as downspike functions. We will identify a downspike B_o-function with a train of negative pulses, and the upspike B_o-function with a train of positive pulses.

The flip-flop circuit, given in Diagram 4 will approximately analyse (5.4) where $\beta(t)$ and ${}_o\beta'(t)$ are a train of negative pulses (approximate in the sense that a pulse is never of zero width in duration and that a flip flop cannot be triggered in zero time).

Diagram 4

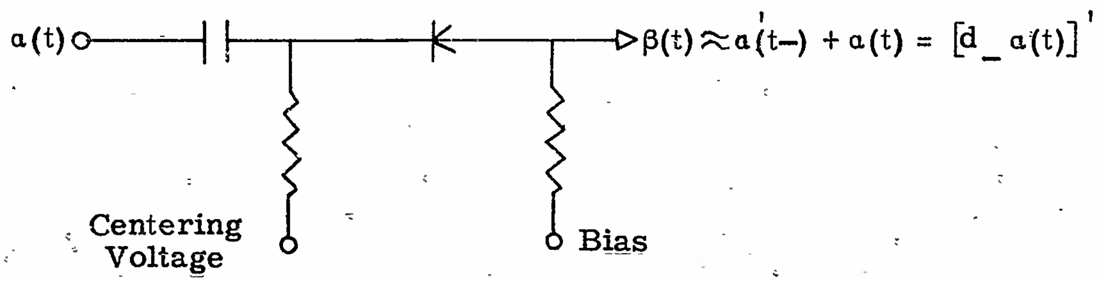


The circuit given in Diagram 5 where $\alpha(t)$ is a $B_{T/2}(o)$-function produces approximately

$$[\alpha\_\ \alpha(t)]' = \alpha'(t-) + \alpha(t)$$

or negative pulses at the down jump points of $\alpha(t)$ (the pulses are negative with respect to the Bias voltage). If $E(t)$ is the clock function, defined in Definition 1.8 of Part I, then by the reasoning, leading to Theorem 1.7 of Part I, Diagrams 3, 4 and 5

Diagram 5

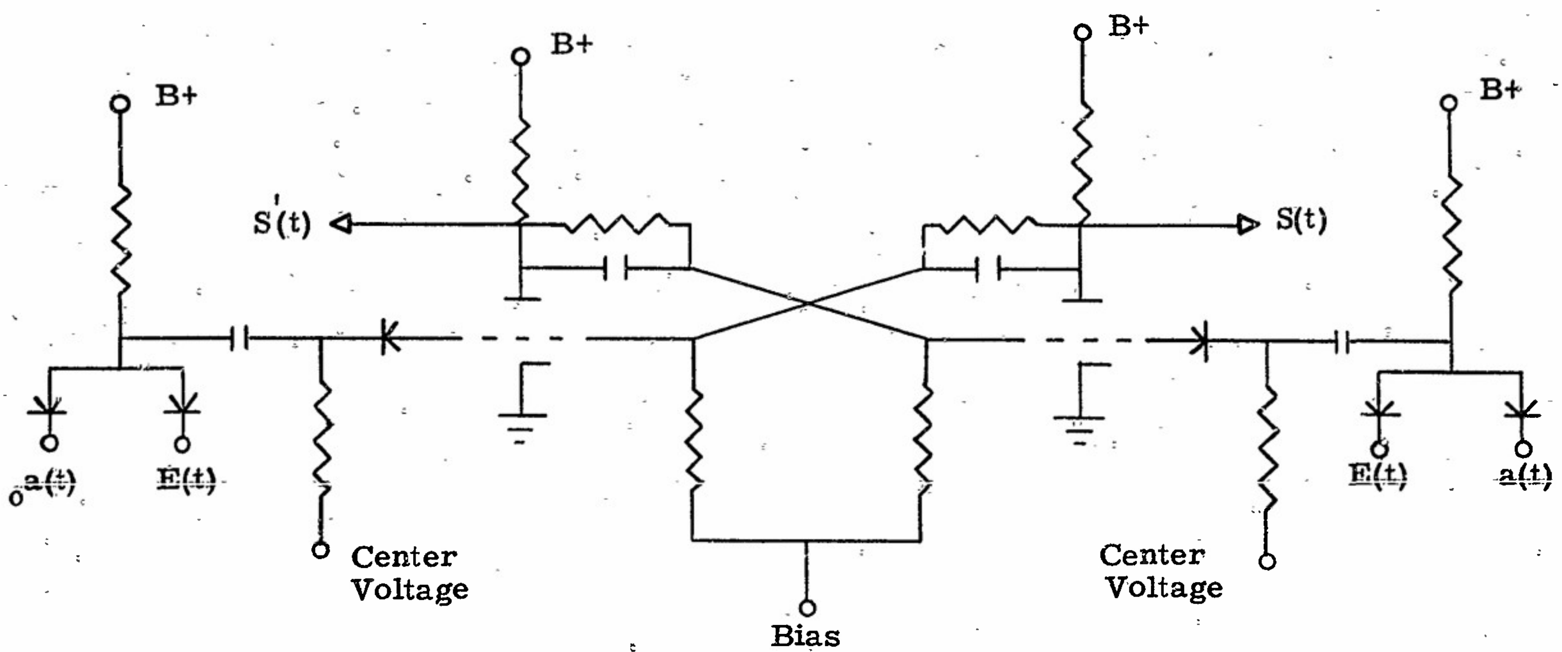


the circuit of Diagram 6 will analyse the difference equation

$$S(t+\tau) = a(t)\,S'(t) + {}_{o}a(t)\,S(t) \qquad (5.3)$$

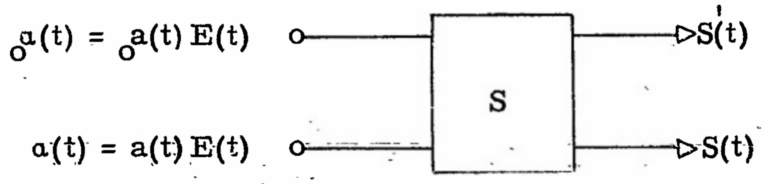
where $a(t)$ and ${}_{o}a(t)$ are $B_{\tau}(o)$-functions. It will be convenient for our purposes to

Diagram 6



regard the flip flop of Diagram 6 as a box with two inputs, $\alpha(t) = a(t)E(t)$ and ${}_{o}\alpha(t) = {}_{o}a(t)E(t)$ and two outputs, $S(t)$ and $S'(t)$. This representation is shown in Diagram 7. There are other circuits and devices which may be used to analyse

Diagram 7



(5.3) other than the flip flop shown in Diagram 6. However, it will not be our present purpose to give this further attention.

Let us now consider the design of a trinary counter. Following the notation of (3.6), let

$$E^0(t_i) = S_1'(t_i)\,S_2'(t_i)$$

$$E^1(t_i) = S_1'(t_i)\,S_2(t_i)$$

$$E^2(t_i) = S_1(t_i)\,S_2'(t_i)$$

$$E^3(t_i) = S_1(t_i)\,S_2(t_i) \tag{5.4}$$

Let $B(t_i)$ be the function of t_i that is to be counted. If $B(t_i) = I$ and $E^2(t_i) = I$ or $E^3(t_i) = I$, then let $E^0(t_{i+1}) = I$, $E^1(t_{i+2})$, $E^2(t_{i+3}) = I$, $E^0(t_{i+4}) = I$ etc. until $B(t_j) = 0$ some $j > i$. If $B(t_i) = I$ and $E^1(t_i) = I$, then let $E^2(t_{i+1}) = I$, $E^1(t_{i+3}) = I$ etc. until $B(t_j) = 0$ some $j > i$. If $B(t_i) = I$ and $E^0(t_i) = I$, then let $E^1(t_{i+1}) = I$, $E^2(t_{i+2}) = I$, $E^0(t_{i+3}) = I$, $E^1(t_{i+4}) = I$ etc. until $B(t_j) = 0$ some $j > i$. If $B(t_i) = 0$, let $E^0(t_i) = E^0(t_{i+1})$, $E^1(t_i) = E^1(t_{i+1})$, $E^2(t_i) = E^2(t_{i+1})$ and $E^3(t_i) = E^3(t_{i+1})$. It is clear that these conditions are the conditions for a counter which counts when $B(t_i) = I$, cycles on three when counting and stops counting when $B(t_i) = 0$. Table 3 shows the operation of this counter more explicitly.

Table 3

	$E^0(t_{i+1})$	$E^1(t_{i+1})$	$E^2(t_{i+1})$	$E^3(t_{i+1})$
$E^0(t_i)\ B^1(t_i)$	I	0	0	0
$E^1(t_i)\ B^1(t_i)$	0	I	0	0
$E^2(t_i)\ B^1(t_i)$	0	0	I	0
$E^3(t_i)\ B^1(t_i)$	0	0	0	I
$E^0(t_i)\ B(t_i)$	0	I	0	0
$E^1(t_i)\ B(t_i)$	0	0	I	0
$E^2(t_i)\ B(t_i)$	I	0	0	0
$E^3(t_i)\ B(t_i)$	I	0	0	0

From Table 3 and the preceding discussion the canonical form of the trinary counter is

$$
\begin{aligned}
E^0(t_{i+1}) &= |E^2(t_i) + E^3(t_i)|\ B(t_i) + E^0(t_i)\ B^1(t_i) \\
E^1(t_{i+1}) &= E^0(t_i)\ B(t_i) + E^1(t_i)\ B^1(t_i) \\
E^2(t_{i+1}) &= E^1(t_i)\ B(t_i) + E^2(t_i)\ B^1(t_i) \\
E^3(t_{i+1}) &= E^3(t_i)\ B^1(t_i)\ .
\end{aligned}
\tag{5.5}
$$

By (5.5) we have

$$E^0(t_i) + E^1(t_i) = S_1'(t_i)$$

$$E^0(t_i) + E^2(t_i) = S_2'(t_i)$$

$$E^0(t_i) + E^3(t_i) = S_1'(t_i)\ S_2'(t_i) + S_1(t_i)\ S_2(t_i)$$

$$E^1(t_i) + E^2(t_i) = S_1(t_i)\ S_2'(t_i) + S_1'(t_i)\ S_2(t_i)$$

$$E^1(t_i) + E^3(t_i) = S_2(t_i)$$

$$E^2(t_i) + E^3(t_i) = S_1(t_i) \tag{5.6}$$

If we combine (5.5) and (5.6), we obtain

$$S_2(t_{i+1}) = [S_1'(t_i)\ B(t_i)]\ S_2'(t_i) + B^1(t_i)\ S_2(t_i)$$

$$S_1(t_i) = [S_2(t_i)\ B(t_i)]\ S_1'(t_i) + B^1(t_i)\ S_1(t_i)\quad . \tag{5.7}$$

The equations of (5.7) are in the form of (5.3) where t_i $(i = 0, 1, 2...)$ are the down jump points of the clock function $E(t)$. The equations of (5.7) are clearly equivalent to

$$S_2(t_{i+1}) = [S_1'(t_i)\ S_2'(t_i)\ B(t_i)]\ S_2'(t_i) + [S_2(t_i)\ B(t_i)]'\ S_2(t_i)$$

$$S_1(t_{i+1}) = [S_1'(t_i)\ S_2(t_i)\ B(t_i)]\ S_1'(t_i) + [S_1(t_i)\ B(t_i)]'\ S_1(t_i)\quad . \tag{5.8}$$

The redundancy in (5.8) is imposed for the practical purpose of insuring the triggering of the flip flop when $a(t) = {}_oa(t) = I$, some t.

By (5.3), (5.8) and Diagram 6 we will need two flip flops S_1 and S_2 with respective inputs $a_1(t)$, ${}_oa_1(t)$ and $a_2(t)$, ${}_oa_2(t)$ where

$$a_1(t) = S_1'(t_i)\ S_2(t_i)\ B(t)\quad ,\quad {}_oa_1(t) = S_1(t)\ B(t)$$

$$a_2(t) = S_1'(t)\ S_2(t)\ B(t)\quad ,\quad {}_oa_2(t) = S_2(t)\ B(t)\quad . \tag{5.9}$$

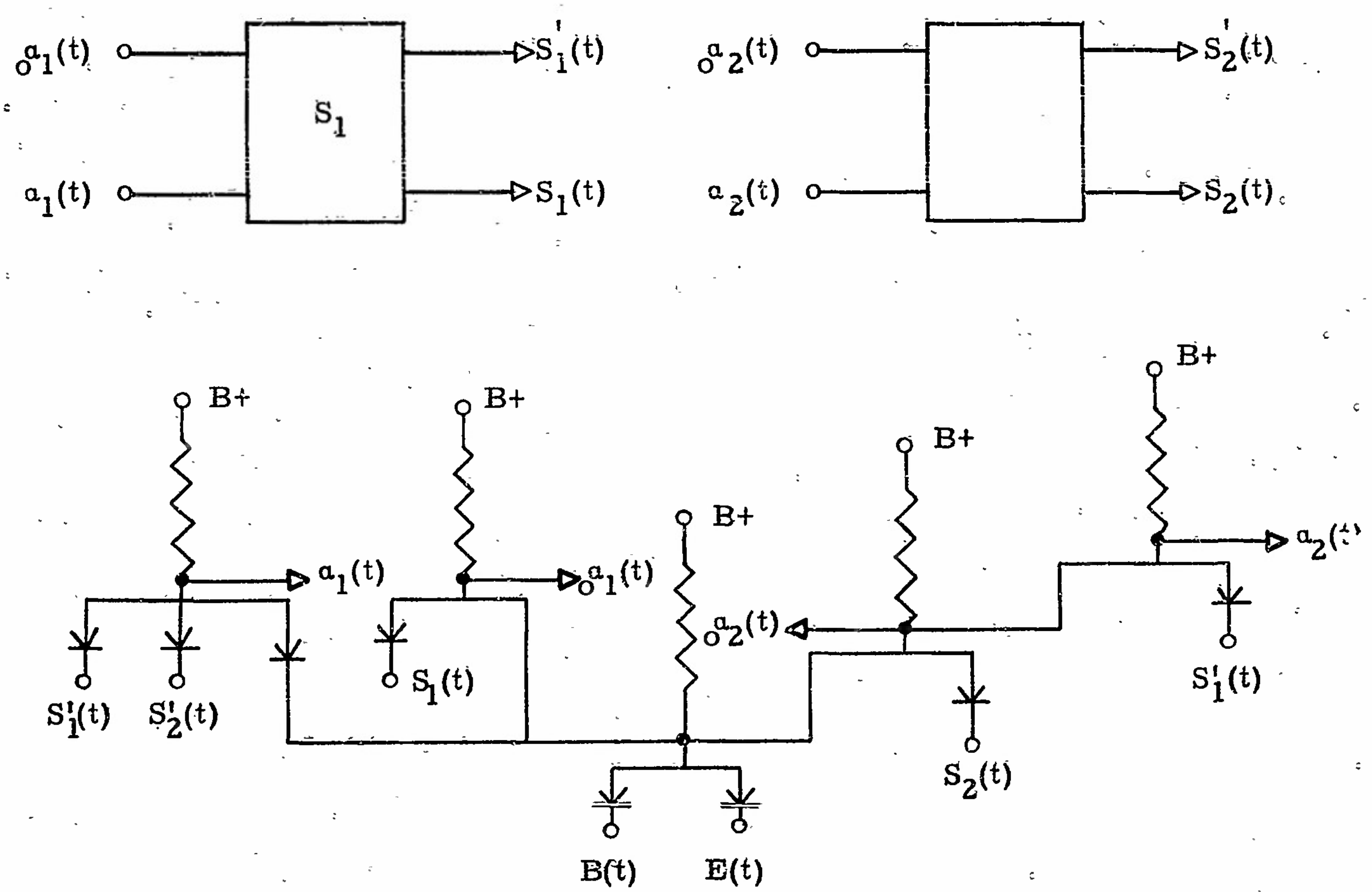
By (5.9) the $\alpha_1(t)$, ${}_0\alpha_1(t)$ and ${}_0\alpha_2(t)$ inputs, as defined by Diagram 7, are

$$\alpha_1(t) = S_1'(t)\, S_2'(t)\, B(t)\, E(t) \quad , \quad {}_0\alpha_1(t) = S_1(t)\, B(t)\, E(t)$$

$$\alpha_2(t) = S_1'(t)\, S_2(t)\, B(t)\, E(t) \quad , \quad {}_0\alpha_2(t) = S_2(t)\, B(t)\, E(t) \quad . \tag{5.10}$$

After a consideration of Diagrams 3 and 7 and (5.10) one sees that Diagram 8 represents the circuit diagram of the above discussed trinary counter or the physical device which will analyse the equation of (5.8).

Diagram 8



The method we have used to obtain the design of Diagram 8 is not necessarily the most efficient design technique. We have used this method because it shows most clearly the nature of a Boolean system. A study of design techniques will be given at a later date.

Let us suppose the probabilities,

$$P(|B(t_i)|) = p \tag{5.11}$$

for (i = 0,1,2...) and

$$q = 1 - p \quad ,$$

and that the event $|B(t_i)|$ is independent of $E^0(t_i)$, $E^1(t_i)$, $E^2(t_i)$, and $E^3(t_i)$ for (i = 0,1,2....). By (4.1) and (5.11) we then have

$$e(t_{i+1}) = \begin{pmatrix} P(|E^0(t_{i+1})|) \\ P(|E^1(t_{i+1})|) \\ P(|E^2(t_{i+1})|) \\ P(|E^3(t_{i+1})|) \end{pmatrix} \begin{pmatrix} q & o & p & p \\ p & q & o & o \\ o & p & q & o \\ o & o & o & q \end{pmatrix} \begin{pmatrix} P(|E^0(t_i)|) \\ P(|E^1(t_i)|) \\ P(|E^2(t_i)|) \\ P(|E^3(t_i)|) \end{pmatrix} = Te(t_i)$$

for (i = 0,1,2,...), where the initial condition is chosen to be $e(t_o)$, as the Markov chain for the trinary counter of Diagram 8. Now the matrix, corresponding to (4.39), will be by (5.12),

$$(I - sT)^{-1} = \begin{pmatrix} 1 - sq & 0 & -sp & -sp \\ -sp & 1 - sq & 0 & 0 \\ 0 & 0 & 1 - sq & 0 \\ 0 & 0 & 0 & 1 - sq \end{pmatrix}^{-1}$$

$$= \frac{1}{|I - sT|} \begin{pmatrix} (1-sq)^3, & (sp)^2(1-sq), & (sp)(1-sq)^2, & (sp)(1-sq)^2 \\ (sp)(1-sq)^2, & (1-sq)^3, & (sp)^2(1-sq), & (sp)^2(1-sq) \\ (sp)^2(1-sq), & (sp)(1-sq)^2, & (1-sq)^3, & (sp)^3 \\ 0, & 0, & 0, & (1-sq)^3 - (sp)^3 \end{pmatrix} \tag{5.13}$$

where $|I - sT|$ is the determinant of $I - sT$, given by

$$|I - sT| = (1 - sq)\,[(1 - sq)^3 - (sp)^3] \quad . \tag{5.14}$$

The roots of $|I - sT|$ are

$$\alpha_1 = 1, \ \alpha_2 = \frac{1}{q}, \ \alpha_3 = \frac{-(1 - 3q) + i\sqrt{3}(q + 1)}{2(p^3 + q^3)} \text{ and } \alpha_4 = \frac{-(1 - 3q) - i\sqrt{3}(q + 1)}{2(p^3 + q^3)} \tag{5.15}$$

where $i = \sqrt{-1}$. If $1 > q > 0$, then $S_2 > 1$ and

$$|\alpha_3| = |\alpha_4| = \frac{\sqrt{1 + 3q^2}}{1 - 3q + 3q^2} > 1 \quad .$$

Thus $\alpha_1 = 1$ is the only root on the unit circle $|s| = 1$. By (4.45) and (5.13) we obtain

$$A_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.16}$$

as the matrix associated with the root $\alpha_1 = 1$. Combining (5.16) with (4.46) we have

$$\lim_{j \to \infty} T^i = A_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

and thus by (5.12) we obtain

$$\lim_{i \to \infty} P(|E^j(t_i)|) = P(|E^j(\infty)|) = \frac{1}{3} \quad \text{for } (j = 0,1,2) \text{ and}$$

$$\lim_{i \to \infty} P(|E^3(t_i)|) = P(|E^3(\infty)|) = 0 \quad . \tag{5.17}$$

By (5.17) and (5.6) in conjunction with the rules of probability we have finally

$$P(|S_1(\infty)|) = P(|S_2(\infty)|) = \frac{1}{3}$$

and

$$P(|S_1'(\infty)|) = P(|S_2'(\infty)|) = \frac{2}{3}$$

as would be expected intuitively.

If $P(|B^1(t_i)|) = q = 0$ for $(i = 0,1,2,\ldots)$, then the matrix T of (5.12) becomes

$$T_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.18}$$

and

$$|I - sT_1| = (1 - s^3)$$

with roots

$$\alpha_1 = 1, \ \alpha_2 = e^{2i\pi/3} \qquad s_3 = e^{4i\pi/3} \quad ,$$

the cube roots of unity. It is not difficult to show in this case by (4.45) that

$$\lim_{N \to \infty} \sum_{j=1}^{N} \frac{T_1^j}{N} = A_1 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.19}$$

thereby demonstrating the validity of (4.47). (5.19) also follows directly from (5.18) by multiplication and summing by noticing that

$$T_1^n = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = T_1 \quad \text{if } n \equiv 1 \ \text{Mod } 3$$

$$T_1^n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{if } n \equiv 2 \text{ Mod } 3$$

$$T_1^n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{if } n \equiv 0 \text{ Mod } 3$$

A further study of matrices of the above type will be made in a later report. One should notice that if $p = 0$, then the matrix T of (5.12) becomes I, the identity matrix. This corresponds to the counter locking on its initial configuration.

In conclusion of this section let us note the following relationship between the probabilities of $B_T(o)$ functions and the current drain and the power consumed in an "and" gate. Let $V(A) = E_H$ if $A = 1$ and $V(A) = E_L$ if $A = 0$.

Diagram 9

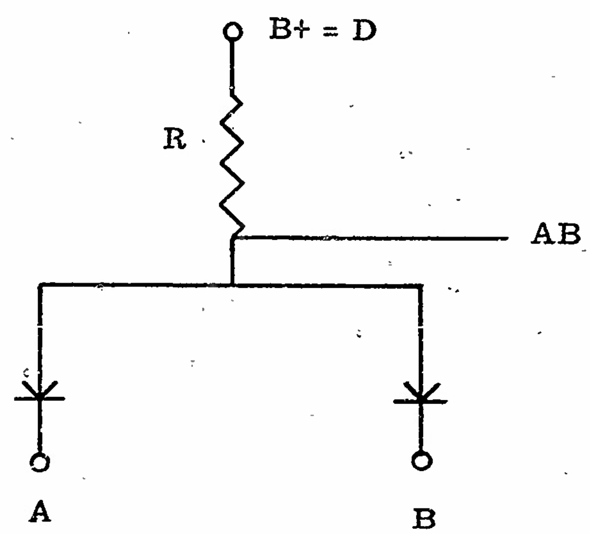


Table 4

V(A)	V(B)	V(AB)	Current through R	Voltage across R	Power
E_L	E_L	E_L	$D - E_L/R$	$D - E_L$	$\frac{(D - E_L)^2}{R}$
E_L	E_H	E_L	$D - E_L/R$	$D - E_L$	$\frac{(D - E_L)^2}{R}$
E_H	E_L	E_L	$D - E_L/R$	$D - E_L$	$\frac{(D - E_L)^2}{R}$
E_H	E_H	E_H	$D - E_L/R$	$D - E_H$	$\frac{(D - E_H)^2}{R}$

By Diagram 9 and Table 4 we have

$$\text{Average power dissipated by } R = \frac{(D - E_H)^2}{R} P(|AB|) + \frac{(D - E_L)^2}{R} [1 - P(|AB|)]$$

$$\text{Average current through } R = \frac{D - E_H}{R} P(AB) + \frac{D - E_L}{R} [1 - P(|AB|)]$$

A similar result can be obtained for the "or" gate. The study of the Boolean machine as a Markov chain thereby appears to be of practical significance in the design problems of these machines.